" $\Leftarrow$ " If $\dim M = 0$, then f is constant.

Assume $\dim M > 2$. Then

$$0 = d(f\omega) = df \wedge \omega \quad \text{but} \quad \Omega^1(M) \xrightarrow{\wedge\omega} \Omega^3(M) \text{ is}$$

injective by Ex.1. since $1 \leq \frac{\dim M}{2} - 1$.

So $df = 0$ i.e. f is locally constant.

Since ~~M~~ M is connected, f is constant. $\square$

<u>Ex.3.</u> Consider $S^2 \times S^4$. The de Rham cohomology reads

$$H^k_{dR}(S^2 \times S^4) = \begin{cases} \mathbb{R} & k = 0, 2, 4, 6 \\ 0 & \text{else} \end{cases} \qquad (*)$$

which is also the cohomology of $\mathbb{CP}^3$. In fact, if $\alpha \in \Omega^2(S^2)$ and $\beta \in \Omega^4(S^4)$ are volume forms, then the cohomology $(*)$ is generated by

$$1, \quad \pi_1^*\alpha, \quad \pi_2^*\beta, \quad \pi_1^*\alpha \wedge \pi_2^*\beta$$
$$(\deg \quad 0 \qquad 2 \qquad 4 \qquad\qquad 6 \quad)$$

this follows from the socalled Künneth theorem.

(Here π_1, π_2 are the projections onto the factors)

Now we see that $S^2 \times S^4$ is not cohomologically symplectic, since the only candidate for a symplectic form form ω such that $[\omega] = [\pi_1^*\alpha]$ up to scalar multiple would be a ~~scalar~~ ~~multiple~~ but

$$[\omega]^2 = [\pi_1^*\alpha] \cup [\pi_1^*\alpha] = [\pi_1^*\alpha \wedge \pi_1^*\alpha] = [\pi_1^*(\alpha \wedge \alpha)] = \pi_1^*([\alpha \wedge \alpha])$$

is zero, since $[\alpha \wedge \alpha] = [\alpha]^2 = 0$ in $H^*_{dR}(S^2)$ for degree reasons (in fact, spheres never have non-trivial cup-products) $\square$

Ex. 4.

(1) $\Omega_\lambda = \lambda\, \pi_1^*(\omega_1) + \lambda^{-1} \pi_2^*(\omega_2)$ is symplectic

closed: d commutes with pullback and ω_1, ω_2 are closed so we get

$$d\Omega_\lambda = \lambda\, \pi_1^* d\omega_1 + \lambda^{-1} \pi_2^* d\omega_2 = 0.$$

non-deg. $$\Omega_\lambda \wedge \Omega_\lambda = \sum_{i=1}^{2} \lambda^{\pm 2} \underbrace{\pi_i^* \omega_i \wedge \pi_i^* \omega_i}_{\begin{gathered} \parallel \\ \pi_i^*(\underbrace{\omega_i \wedge \omega_i}_{=0 \text{ for degree reasons}}) \end{gathered}} + 2\, \pi_1^* \omega_1 \wedge \pi_2^* \omega_2$$

$$= 2\, \pi_1^* \omega_1 \wedge \pi_2^* \omega_2$$

which is a volume form for $\Sigma_1 \times \Sigma_2$.

We also see that $\Omega_\lambda \wedge \Omega_\lambda$ does not depend on λ.

(2) Need to check $\Omega_\lambda |_{a \times b} \equiv 0$.

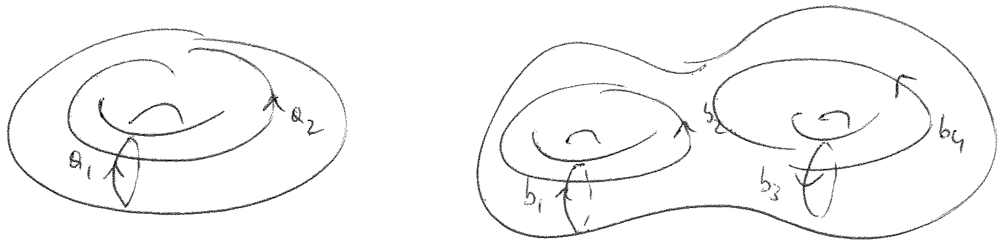
this follows from

$$\begin{tikzcd} T_{(x,y)}(a \times b) \times T_{(x,y)}(a \times b) \arrow[r, "\pi_1^* \omega_1"] \arrow[d, "D\pi_1 \times D\pi_1"] & \mathbb{R} \\ \underbrace{T_x a}_{\text{1-dim}} \times T_x a \arrow[ur, bend right, "\omega_1"'] & \end{tikzcd}$$

1-dim $\Rightarrow$ Lagrangian for ω_1 $\Rightarrow$ $\pi_1^* \omega_1 |_{a \times b} \equiv 0$.

and similarly for $\pi_2^* \omega_2$.

(3)



$$H_2(\Sigma_1\times\Sigma_2;\mathbb{Z}) \cong \mathbb{Z}\left\{\Sigma_1\times\{pt\},\ \{pt\}\times\Sigma_2,\ a_i\times b_j \;\middle|\; \begin{array}{l} i=1\ldots 2g\\ j=1\ldots 2h\end{array}\right\}$$

The homomorphism $P_\lambda : H_2(\Sigma_1\times\Sigma_2;\mathbb{Z}) \to \mathbb{R}$

is determined by what it does to generators:

- $\Sigma_1\times\{pt\} = i_1(\Sigma_1) \mapsto \int_{i_1(\Sigma_1)} \Omega_\lambda = \int_{\Sigma_1} i_1^*\Omega_\lambda$

$$\begin{array}{ccc} \Sigma_1 & \xhookrightarrow{i_1} & \Sigma_1\times\Sigma_2 \\ x & \longmapsto & (x,pt) \end{array}$$

$$= \int_{\Sigma_1}\left(\lambda(\pi_1 i_1)^*(\omega_1) + \cancel{\lambda^{-1}(\pi_2 i_1)^*(\omega_2)}\right)$$

$\pi_2 i_1 = \text{const}$

$$\begin{tikzcd} \Sigma_1 \arrow[r, hook, "i_1"] \arrow[dr, "id"'] & \Sigma_1\times\Sigma_2 \arrow[d, "\pi_1"] \\ & \Sigma_1 \end{tikzcd}$$

$$= \int_{\Sigma_1} \lambda\,\omega_1$$

$$\begin{tikzcd} \Sigma_1 \arrow[r, hook, "i_1"] \arrow[dr, "\text{constant at } pt"'] & \Sigma_1\times\Sigma_2 \arrow[d, "\pi_2"] \\ & \Sigma_2 \end{tikzcd}$$

$$= \lambda r_1 .$$

- $pt\times\Sigma_2 \overset{P_\lambda}{\longmapsto} \lambda^{-1} r_2$.

- $a_i\times b_j \overset{P_\lambda}{\longmapsto} 0$ by (2).

$\Rightarrow$ $\mathrm{im}(P_\lambda) = \mathbb{Z}\lambda r_1 + \mathbb{Z}\lambda^{-1} r_2 \subseteq \mathbb{R}$ (a discrete subgroup)

(4) $\varphi : (\Sigma_1 \times \Sigma_2, \Omega_1) \to (\Sigma_1 \times \Sigma_2, \Omega_\lambda)$ a symplectomorphism.

Since φ is a diffeomorphism,

$$\varphi_* : H_2(\Sigma_1 \times \Sigma_2; \mathbb{Z}) \to H_2(\Sigma_1 \times \Sigma_2; \mathbb{Z})$$

is an isomorphism, hence

$$\left\{ \varphi(\Sigma_1 \times pt),\ \varphi(pt \times \Sigma_2),\ \varphi(a_i \times b_j) \;\middle|\; \begin{matrix} i = 1 \ldots 2g \\ j = 1 \ldots 2h \end{matrix} \right\}$$

also generate $H_2(\Sigma_1 \times \Sigma_2; \mathbb{Z})$.

Now $$\int_{\varphi(A)} \Omega_\lambda = \int_A \varphi^* \Omega_\lambda = \int_A \Omega_1 .$$

So $\operatorname{im}(p_\lambda) = \operatorname{im}(p_1)$, i.e.

$$\mathbb{Z} \lambda r_1 + \mathbb{Z} \lambda^{-1} r_2 = \mathbb{Z} \cdot r_1 + \mathbb{Z} r_2$$

in particular, $\lambda r_1 = k \cdot r_1 + l \cdot r_2$ for some $k, l \in \mathbb{Z}$.

$$\Rightarrow \quad \lambda \in \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot r_2 / r_1$$

Since this is a discrete subset of $\mathbb{R}$, a generic choice of λ lies outside, and for such λ there is no symplectomorphism. $\square$