If $f : E \to F$ is a fibrewise linear map of vector bundles over X, $\ker(f)$ need not be a subbundle of E!

E.g. take the Möbius bundle $M \subseteq S^1 \times \mathbb{R}^2$ parametrized such that the fibre over $e^{i\varphi} \in S^1$, $\varphi \in (0, 2\pi]$ is the line $\{ r e^{i\varphi/2} \mid r \in \mathbb{R} \} \subseteq \mathbb{C} = \mathbb{R}^2$.

Then the composite

$$\begin{tikzcd} M \arrow[r, hook, "incl"] \arrow[rr, bend right, "f"'] & S^1 \times \mathbb{R}^2 \arrow[r, "id \times pr_2"] & S^1 \times \mathbb{R} \end{tikzcd}$$

where $pr_2 : \mathbb{R}^2 \to \mathbb{R}$ is projection onto the second factor is smooth and linear on every fibre but

$$\dim(\ker(f)_x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

fibre over $x \in S^1$

So $\ker(f)$ is not a locally trivial bundle, in fact not a manifold.