(Ex. 4.)

(1) must check

$$\begin{cases} \gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2g(v,w)\,\mathrm{id} \\ \gamma(v)^{+} = -\gamma(v) \end{cases} \qquad \forall v,w \in H.$$

note the first is equivalent to

$$\gamma(v)^2 = -\|v\|^2\,\mathrm{id} \qquad \forall v \in H.$$

let $\phi, \psi \in V^{0,1}$ be an ONB for $h(-,-)$.

then $1, \phi, \psi, \phi\wedge\psi$ is ONB for $\Lambda^* V^{0,1}$.

$$\boxed{\gamma(v)^2 \cdot 1 = -\|v\|^2 \cdot 1}$$

$$\underset{(1,0)}{\gamma(v)^2 \cdot 1} = \gamma(v)\ \sqrt{2}\, v^{0,1}$$

$$= 2\cdot\Big(-*(v^{1,0}\wedge * v^{0,1}),\ \underbrace{v^{0,1}\wedge v^{0,1}}_{=0}\Big)$$

$$= -2 * (\underbrace{v^{1,0}\wedge * v^{0,1}}_{= v^{1,0}\wedge * \overline{v^{1,0}} = h(v^{1,0}, v^{1,0})\,\mathrm{vol}})$$

$$= -2\, h(v^{1,0}, v^{1,0})\cdot 1$$

using $v^{1,0} = \frac{1}{2}(v - iJ(v))$ we get

$$h(v^{1,0}, v^{1,0}) = \frac{1}{4}\Big(\overbrace{h(v,v)}^{\|v\|^2} + \overbrace{h(-iJv,-iJv)}^{\|v\|^2} + \overbrace{h(v,-iJv) + h(-iJv,v)}^{=0 \text{ by symmetry}}\Big)$$

$$= \frac{1}{2}\|v\|^2. \qquad \Rightarrow \text{claim.}$$

write $\beta := \phi \wedge \psi$ $\leadsto$ $\bar\beta = \bar\phi \wedge \bar\psi$ is ONB for $V^{2,0}$

$*\bar\beta = \lambda \cdot \bar\beta$, $\lambda \in S^1$ since $*$ is an isometry

$$\boxed{\gamma(v)^2 \beta = -\|v\|^2 \beta}$$

$$\gamma(v)^2\beta = \gamma(v) \cdot \left(-\sqrt{2} * (v^{1,0} \wedge * \beta)\right)$$

$$= -2 \Big(\underbrace{- * (v^{1,0} \wedge \underbrace{*}_{\pm 1} (* (v^{1,0} \wedge * \beta)))}_{= 0 \text{ since } v^{1,0} \wedge v^{1,0} = 0}, \; v^{0,1} \wedge * (v^{1,0} \wedge * \beta) \Big)$$

$$= -2 \, v^{0,1} \wedge * (v^{1,0} \wedge * \beta) = \lambda \beta \quad \text{for some } \lambda \in \mathbb{C}.$$

$$\lambda \, \mathrm{vol} = h\left(-2 v^{0,1} \wedge * (v^{1,0} \wedge * \beta), \beta\right) \mathrm{vol}$$

$$= -2 \, v^{0,1} \wedge * (v^{1,0} \wedge * \beta) \wedge * \bar\beta$$

$$= -2 \left(v^{0,1} \wedge * \bar\beta\right) \wedge * \overline{\left(v^{0,1} \wedge * \bar\beta\right)}$$

$$= -2 \, h\left(v^{0,1} \wedge * \bar\beta, v^{0,1} \wedge * \bar\beta\right) \mathrm{vol}$$

$v^{0,1} \perp * \bar\beta$ & $\|\bar\beta\| = 1$

$$= -2 \, h(v^{0,1}, v^{0,1}) \cdot \; \mathrm{vol}$$

$$= -\|v\|^2 \, \mathrm{vol}. \quad \text{as before.}$$

$\Rightarrow$ claim.

$\phi, \psi \in V^{0,1}$ ONB $\rightsquigarrow$ $\bar\phi, \bar\psi \in V^{1,0}$ ONB

$v^{0,1} = a \cdot \phi + b \cdot \psi$, $v^{1,0} = \bar a \bar\phi + \bar b \bar\psi$

for some $a, b \in \mathbb{C}$

$\boxed{\gamma(v)^2 \phi = -\|v\|^2 \phi}$ (by symmetry the same holds for ψ).

$\gamma(v)^2 \phi = \lambda \phi + \mu \psi$, $\lambda, \mu \in \mathbb{C}$.

$-\frac{1}{2} \langle \gamma(v)^2 \phi, \psi \rangle \cdot \mathrm{vol}$

$$= \langle v^{0,1} \wedge * \underbrace{(v^{1,0} \wedge * \phi)}_{= \langle v^{1,0}, \bar\phi \rangle \mathrm{vol} = \bar a \,\mathrm{vol}}, \psi \rangle \mathrm{vol} + \underbrace{\langle *(v^{1,0} \wedge *(v^{0,1} \wedge \phi)), \psi \rangle \mathrm{vol}}_{= \overline{\langle \psi, *(v^{1,0} \wedge *(v^{0,1} \wedge \phi)) \rangle}}$$

$$= \bar a \cdot \langle v^{0,1}, \psi \rangle \mathrm{vol} + \overline{\overline{\psi \wedge \underbrace{**}_{=-1} (v^{1,0} \wedge *(v^{0,1} \wedge \phi))}}$$

$$= \bar a b \,\mathrm{vol} + \underbrace{(v^{1,0} \wedge \bar\psi) \wedge *(v^{0,1} \wedge \phi)}$$

$$= v^{1,0} \wedge \bar\psi \wedge *\overline{(v^{1,0} \wedge \bar\phi)}$$

$$= \langle v^{1,0} \wedge \bar\psi, v^{1,0} \wedge \bar\phi \rangle \mathrm{vol}$$

$$= \langle \bar a \bar\phi \wedge \bar\psi, \bar b \bar\psi \wedge \bar\phi \rangle \mathrm{vol}$$

$$= -\bar a \cdot b \langle \bar\phi \wedge \bar\psi, \bar\phi \wedge \bar\psi \rangle \mathrm{vol}$$

$$= -\bar a \cdot b \,\mathrm{vol}$$

$= 0. \implies \mu = 0.$

with ϕ instead of φ we get

$$-\frac{1}{2}\langle \gamma(v)^2\phi, \phi\rangle \, vol = \|a\|^2 \, vol + \langle \bar{b}\ \bar{\varphi}\wedge\bar{\phi}, \bar{b}\ \bar{\varphi}\wedge\bar{\phi}\rangle \, vol$$

$$= (\|a\|^2 + \|b\|^2) . \, vol$$

$$= \|v^{0,1}\|^2 \, vol$$

$$= \frac{1}{2}\|v\|^2 \, vol .$$

$$\Rightarrow \quad \lambda = \frac{1}{2}\|v\|^2 .$$

$$\Rightarrow \quad \gamma(v)^2\phi = -\|v\|^2\phi .$$

$$\boxed{\gamma(v)^{\dagger}\cdot 1 = -\gamma(v)\cdot 1}$$

note that both sides are orthogonal to V_+

e.g. $\forall w \in V_+$: $\quad \langle \gamma(v)^{\dagger}\cdot 1, w\rangle = \langle \underbrace{1}_{\in V_+}, \underbrace{\gamma(v)\cdot w}_{\in V_-}\rangle = 0 .$

so we only need to check that $\langle \gamma(v)^{\dagger}\cdot 1, \phi\rangle = \langle -\gamma(v)\cdot 1, \phi\rangle$ for all $\phi \in V^{0,1}$.

now $\quad \langle \gamma(v)^{\dagger}\cdot 1, \phi\rangle^{vol} = \langle 1, \gamma(v)\phi\rangle \, vol$

$$= \langle 1, \sqrt{2}\,(-*(v^{1,0}\wedge *\phi), v^{0,1}\wedge\phi)\rangle \, vol$$

$$= -\sqrt{2}\, \underbrace{**}_{=+1}\, \overline{(v^{1,0}\wedge *\phi)}$$

$$= -\sqrt{2}\, v^{0,1}\wedge *\bar{\phi} .$$

$$-\langle \gamma(v)\cdot 1, \phi\rangle \, vol = -\langle \sqrt{2}\, v^{0,1}, \phi\rangle \, vol = -\sqrt{2}\, v^{0,1}\wedge *\bar{\phi} .$$

$\Rightarrow$ claim.

$$\boxed{\gamma(v)^\dagger \phi = - \gamma(v) \cdot \phi}$$

both sides orthogonal to V_-, so suffices to check inner products with 1 and $\beta = \phi \wedge \psi$.

$$\langle \gamma(v)^\dagger \phi, 1 \rangle \, vol = \langle \phi, \sqrt{2}\, v^{0,1} \rangle \, vol$$

$$= \sqrt{2}\, v^{1,0} \wedge * \phi$$

$$\langle -\gamma(v) \phi, 1 \rangle \, vol = \sqrt{2} \langle *(v^{1,0} \wedge * \phi), 1 \rangle \, vol$$

$$= \sqrt{2}\, \underbrace{**}_{=+1} (v^{1,0} \wedge * \phi)$$

$\Rightarrow$ agree.

$$\langle \gamma(v)^\dagger \phi, \beta \rangle \, vol = \langle \phi, \gamma(v) \beta \rangle \, vol$$

$$= \langle \phi, -\sqrt{2} * (v^{1,0} \wedge * \beta) \rangle \, vol$$

$$= -\sqrt{2}\, \phi \wedge \underbrace{**}_{-1} (v^{0,1} \wedge * \bar{\beta})$$

$$= \sqrt{2}\, \phi \wedge v^{0,1} \wedge * \bar{\beta}.$$

$$- \langle \gamma(v) \phi, \beta \rangle \, vol = -\sqrt{2} \langle v^{0,1} \wedge \phi, \beta \rangle \, vol$$

$$= -\sqrt{2}\, \underbrace{v^{0,1} \wedge \phi}_{= -\phi \wedge v^{0,1}} \wedge * \bar{\beta}$$

$\Rightarrow$ agree.

$\Rightarrow$ claim.

$$\boxed{\gamma(v)^\dagger \rho = -\gamma(v)\rho}$$

only need to check inner product with $\phi \in V^{0,1}$.

$$\begin{aligned}
\langle \gamma(v)^\dagger \rho, \phi \rangle \, vol &= \langle \rho, \gamma(v)\phi \rangle \, vol \\
&= \langle \rho, \sqrt{2}\, v^{0,1} \wedge \phi \rangle \, vol \\
&= \sqrt{2}\, v^{1,0} \wedge \bar{\phi} \wedge * \rho
\end{aligned}$$

$$\begin{aligned}
-\langle \gamma(v)\rho, \phi \rangle \, vol &= + \sqrt{2} \langle *(v^{1,0} \wedge * \rho), \phi \rangle \, vol \\
&= \sqrt{2}\, \bar{\phi} \wedge \underbrace{**}_{=-1} (v^{1,0} \wedge * \rho) \\
&= \sqrt{2}\, v^{1,0} \wedge \bar{\phi} \wedge * \rho .
\end{aligned}$$

$\Rightarrow$ claim.

$\square$

~~(2) using the Clifford multiplication of (1) fibrewise gives us a $Spin^c$ - structure on X~~

~~the characteristic line bundle is~~

~~$L = \det(V_\pm) = V^{0,2}$ where $V = TX \otimes_{\mathbb{R}} \mathbb{C}$~~

~~Notice that $V^{1,0} \cong (H, J)$ as complex v.space~~

~~$\leftarrow$ H equipped with the complex vector space structure induced by J, i.e. $i \cdot v := J(v)$.~~

~~and $V^{0,1} \cong \overline{V^{1,0}}$.~~

~~so equivalently the characteristic line bundle of the canonical Spin^c-structure is the determinant bundle of (H, J) (TX, J).~~

Using the Clifford multiplication of (1) fibrewise, we define a Spin^c-structure

$$TX^* \longrightarrow \mathrm{End}(V)$$

where $V := TX^* \otimes_{\mathbb{R}} \mathbb{C}$

The complex structure $J : TX \to TX$ induces the dual one on TX^*, and we split $V \cong V^{1,0} \oplus V^{0,1}$ as in (1) using this complex structure.

The characteristic line bundle is then

$$L_{S_{can}} = \det(V_+) = V^{0,2}$$

the bundle of complex-valued 2-forms of type $(0,2)$.

Recall that the top exterior power of the bundle of $(1,0)$-forms is the <u>canonical bundle</u> K, so $V^{0,2} \cong \overline{V^{2,0}}$ is the <u>anti-canonical</u> bundle: $L_{S_{can}} = K^{-1}$

Equivalently, $K^{-1} = \det(TX, J)$.