Summer term 2022

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Symplectic Geometry

Sheet 9

Exercise 1. Let M be a smooth manifold.

1. Let $\alpha \in \Omega^1(M)$ be a nowhere vanishing 1-form. Show that if α is closed, then the distribution $\ker(\alpha) \subset TM$ is integrable. In fact, show that $\ker(\alpha)$ is integrable if and only if $\alpha \wedge d\alpha = 0$.
2. Let $\omega \in \Omega^2(M)$ be a 2-form of constant rank. Define the distribution $\ker(\omega) \subset TM$ by
$$\ker(\omega)_p = \{X \in T_pM \mid \iota_X \omega_p = 0\}$$
for $p \in M$. Show that $\ker(\omega)$ is integrable if ω is closed.

Exercise 2. Let ∇ be an affine connection on a smooth manifold M. Let $\omega \in \Omega^2(M)$ and suppose that ∇ is compatible with ω, that is, $L_X\omega(Y,Z) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X Z)$ for all $X, Y, Z \in \mathfrak{X}(M)$. Show that
$$d\omega(X,Y,Z) = \omega(T^\nabla(X,Y),Z) - \omega(T^\nabla(X,Z),Y) + \omega(T^\nabla(Y,Z),X)$$
for all vector fields $X, Y, Z \in \mathfrak{X}(M)$.

Exercise 3. Let (M, g, J) be a Kähler manifold and $\mathcal{F}$ a Lagrangian foliation of M such that $T\mathcal{F}$ is preserved by the Levi-Civita connection ∇ of g (this is the set-up of Exercise 3 of Sheet 8). Show that the metric g is flat. That is, show that the curvature tensor of ∇ defined by $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$ vanishes identically for $X, Y, Z \in \mathfrak{X}(M)$.

Exercise 4. On $\mathbb{R}^2$, with its standard coordinates x, y, consider the 2-form
$$\omega = (2 + \sin(2\pi x)\sin(2\pi y))dx \wedge dy\,.$$

1. Show that the complementary foliations $\mathcal{F}$ and $\mathcal{G}$ given by the factors of $\mathbb{R}^2$ define a Künneth structure on $(\mathbb{R}^2, \omega)$.
2. Let ∇ be the Künneth connection of $(\omega, \mathcal{F}, \mathcal{G})$. Calculate the vector fields $\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}$ and $\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial x}$.
3. Let $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$ denote the curvature of ∇. Calculate $R(X,Y)Z$ for suitable vector fields X, Y, Z and show that it does not vanish identically. Hence, the Künneth structure $(\omega, \mathcal{F}, \mathcal{G})$ is not flat.
4. Can you construct a non-flat Künneth structure on the torus T^2?

Please hand in your solutions in the lecture on 15 July 2022.