Summer term 2022

Prof. D. Kotschick
Dr. S. Gritschacher

Symplectic Geometry

Sheet 3

Exercise 1. 1. Let (V, ω) be a $2n$-dimensional symplectic vector space. Prove that for every $l \leq n$ and every $k \leq n - l$ the linear map $\bigwedge^l V^* \xrightarrow{\wedge \omega^k} \bigwedge^{l+2k} V^*$ is injective.

2. Let M be a closed connected $2n$-dimensional manifold and let $\omega \in \Omega^2(M)$ be non-degenerate (but not necessarily closed). Prove that if $k \leq n$ and $k \neq n-1$, then ω^k cannot be exact.

Exercise 2. Let (M, ω) be a connected symplectic manifold and let $f\colon M \to \mathbb{R}\backslash\{0\}$ be a smooth map. Prove that $f\omega$ is symplectic if and only if $\dim(M) = 2$ or f is constant.

Exercise 3. Find a manifold whose cohomology groups are that of $\mathbb{C}P^3$ (and hence that of a symplectic manifold), but which is not (cohomologically) symplectic.

Exercise 4. Let (Σ_1, ω_1) and (Σ_2, ω_2) be closed connected symplectic surfaces. Let $\pi_i\colon \Sigma_1 \times \Sigma_2 \to \Sigma_i$ $(i = 1, 2)$ denote the projection onto the i-th factor.

1. Prove that for every $\lambda \in \mathbb{R}\backslash\{0\}$ the 2-form

$$\Omega_\lambda := \lambda \pi_1^*(\omega_1) + \lambda^{-1} \pi_2^*(\omega_2)$$

is a symplectic form on $\Sigma_1 \times \Sigma_2$ whose associated volume form is independent of λ.

2. Let $a \subset \Sigma_1$ and $b \subset \Sigma_2$ be curves. Show that $a \times b$ is a Lagrangian submanifold of $(\Sigma_1 \times \Sigma_2, \Omega_\lambda)$, and conclude that $\int_{a \times b} \Omega_\lambda = 0$.

3. Suppose that Σ_1 and Σ_2 have genus g and h, respectively. Then $H_2(\Sigma_1 \times \Sigma_2; \mathbb{Z})$ is the free abelian group generated by the fundamental classes of the submanifolds

$$\Sigma_1 \times \mathrm{pt}, \quad \mathrm{pt} \times \Sigma_2, \quad a_i \times b_j \quad (i = 1, \ldots, 2g,\ j = 1 \ldots 2h)$$

where the a_i and b_j are certain curves in Σ_1 and Σ_2, respectively.[1] Integrating Ω_λ over these submanifolds defines a homomorphism $H_2(\Sigma_1 \times \Sigma_2; \mathbb{Z}) \to \mathbb{R}$. Calculate its image.

(please turn)

[1] If you are not familiar with (integral) homology, don't worry. You can still complete this part of the exercise.

4. Let $r_i := \int_{\Sigma_i} \omega_i \in \mathbb{R}$ be the volume of Σ_i. Prove that if $(\Sigma_1 \times \Sigma_2, \Omega_1)$ and $(\Sigma_1 \times \Sigma_2, \Omega_\lambda)$ are symplectomorphic, then there are integers $k, l \in \mathbb{Z}$ with $\lambda = k + l r_2 / r_1$. Conclude that for a generic λ there is no such symplectomorphism.

Please hand in your solutions in the lecture on Friday, 20 May 2022.