LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

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Functional Analysis II

Holiday Extra

Problem A (Measurable functional calculus for commuting self-adjoint operators).
For $n \in \mathbb{N}$ let $T_1, \ldots, T_n$ be pairwise commuting, bounded, self-adjoint operators on a separable Hilbert space $\mathcal{H}$, and let $\mathcal{S} := \sigma(T_1) \times \cdots \times \sigma(T_n) \subset \mathbb{R}^n$. The goal of this exercise is to prove that there exists a unique map $\Phi : \mathcal{M}_b(\mathcal{S}) \to \mathcal{B}(\mathcal{H})$ such that

(*a*) $\Phi(1) = \mathbb{I}$, and $\Phi(\pi_k) = T_k$, where $\pi_k(t_1, \ldots, t_n) := t_k$ for all $k = 1, \ldots, n$.

(*b*) Φ is a linear multiplicative involution.

(*c*) $\|\Phi(f)\| \leqslant \|f\|_\infty$ for all $f \in M_b(\mathcal{S})$, i.e. Φ is continuous.

(*d*) If $(f_k)_k \subset \mathcal{M}_b(\mathcal{S})$ converges pointwise to some $f \in \mathcal{M}_b(\mathcal{S})$ and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n)x \to \Phi(f)x$ for each $x \in \mathcal{H}$.

(*e*) $\Phi(f)S = S\Phi(f)$ for any $S \in \mathcal{B}(\mathcal{H})$ that commutes with $T_1, \ldots, T_n$.

(*i*) Prove that for all Borel sets $A_1, \ldots, A_n \subset \mathbb{R}$, the operators $\chi_{A_1}(T_1), \ldots, \chi_{A_n}(T_n)$ commute, where χ_{A_i} denotes the characteristic function of A_i for $i = 1, \ldots, n$.

(*ii*) Define Φ_0 on step functions f over rectangles, i.e. for functions of the form

$$f = \sum_{i=1}^{N} c_i \, \chi_{A_1^{(i)} \times \cdots \times A_n^{(i)}} , \quad \text{where } A_k^{(i)} \cap A_k^{(j)} = \emptyset \text{ if } i \neq j \ \forall k = 1, \ldots, n$$

for some $N \in \mathbb{N}$, $c_i \in \mathbb{C}$ and Borel sets $A_k^{(i)} \subset \mathbb{R}$ for all $i = 1, \ldots, N$, $k = 1, \ldots, n$, by

$$\Phi_0(f) = f(T_1, \ldots, T_n) := \sum_{i=1}^{N} c_i \, \chi_{A_1^{(i)}}(T_1) \cdots \chi_{A_n^{(i)}}(T_n) .$$

Prove that Φ_0 satisfies the properties (*a*), (*b*), and (*c*) above.

(*iii*) Construct the continuous functional calculus on $\mathcal{S}$.

[*Hint:* Use uniform continuity and part (*ii*).]

(*iv*) Construct the measurable functional calculus described by (*a*) – (*e*) above.

(*v*) Prove that there exists a spectral measure E on $\mathbb{R}^n$ (to be defined) such that

$$\Phi(f) = \int_{\mathcal{S}} f(\lambda_1, \ldots, \lambda_n) \, dE_{(\lambda_1, \ldots, \lambda_n)}$$

for all $f \in \mathcal{M}_b(\mathcal{S})$, in particular, $T_k = \int_{\mathcal{S}} \lambda_k \, dE_{(\lambda_1, \ldots, \lambda_n)}$ for all $k = 1, \ldots, n$.

(*vi*) Prove that there exists a finite measure space (M, Σ, μ), an isometric isomorphism $U : \mathcal{H} \to L^2(M)$, and bounded measurable functions $F_1, \ldots, F_n$ on M, such that

$$(UT_kU^{-1}\varphi)(\xi) = F_k(\xi)\,\varphi(\xi)$$

for μ-almost all $\xi \in M$ and all $k = 1, \ldots, n$.

Problem B (Measurable functional calculus for normal operators).
Let T be a normal operator on a separable Hilbert space $\mathcal{H}$.

(*i*) Prove that there exists a unique map $\Phi : \mathcal{M}_b(\sigma(T)) \to \mathcal{B}(\mathcal{H})$ such that

(*a*) $\Phi(1) = \mathbb{I}$, and $\Phi(z) = T$, where $z : \sigma(T) \to \mathbb{C}, z \mapsto z$.

(*b*) Φ is a linear multiplicative involution.

(*c*) $\|\Phi(f)\| \leqslant \|f\|_\infty$ for all $f \in M_b(\sigma(T))$, i.e. Φ is continuous.

(*d*) If $(f_k)_k \subset \mathcal{M}_b(\sigma(T))$ converges pointwise to $f \in \mathcal{M}_b(\mathcal{S})$ and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n)x \to \Phi(f)x$ for each $x \in \mathcal{H}$.

(*e*) $\Phi(f)S = S\Phi(f)$ for any $S \in \mathcal{B}(\mathcal{H})$ that commutes with T.

[*Hint:* Problem 25.]

(*ii*) There exists a spectral measure G on $\mathbb{C}$ (to be defined) such that

$$\Phi(f) = \int_{\sigma(T)} f(z)\,dG_z$$

for all $f \in \mathcal{M}_b(\sigma(T))$, in particular, $T = \int_{\sigma(T)} z\,dG_z$.

(*iii*) There exists a finite measure space (M, Σ, μ), an isometric isomorphism $U : \mathcal{H} \to L^2(M)$, and a bounded measurable function F on M, such that

$$(UTU^{-1}\varphi)(\xi) = F(\xi)\,\varphi(\xi)$$

for μ-almost all $\xi \in M$.