Prof. T. Ø. Sørensen PhD
S. Gottwald

Winter term 2015/16
Jan 08, 2016

Functional Analysis II

Assignment 11

Problem 41 (Operator convex functions). Let $\mathcal{H}$ be a Hilbert space. A continuous real-valued function f on an interval $I \subset \mathbb{R}$ is called *operator convex* (on I) if for any $\lambda \in [0,1]$ we have $f(\lambda A+(1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$ for any pair $A, B \in \mathcal{B}(\mathcal{H})$ of self-adjoint operators with $\sigma(A) \subset I$ and $\sigma(B) \subset I$. Prove:

(*i*) A continuous real-valued function f is operator convex iff $f(\frac{A+B}{2}) \leqslant \frac{1}{2}(f(A)+f(B))$ for any pair $A, B \in \mathcal{B}(\mathcal{H})$ of self-adjoint operators with spectrum in I.

(*ii*) $f : \mathbb{R} \to \mathbb{R}, t \mapsto t^2$ is operator convex on every interval.

(*iii*) $f : [0,\infty) \to \mathbb{R}, t \mapsto t^3$ is *not* operator convex on $[0,\infty)$.

(*iv*) $f : \mathbb{R} \to \mathbb{R}, t \mapsto |t|$ is *not* operator convex on any interval that contains a neighbourhood of zero.

(*v*) $f : (0,\infty) \to \mathbb{R}, t \mapsto t^{-1}$ is operator convex on $(0,\infty)$.

Problem 42. Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint. Prove:

(*i*) The operator $U(t) := e^{itA}$ constructed via the functional calculus is unitary for all $t \in \mathbb{R}$, and
$$U(t)^* = U(-t)\,, \qquad U(t)U(s) = U(t+s) \quad \forall t, s \in \mathbb{R}\,.$$

(*ii*) The operator-valued function $t \mapsto U(t)$ defined in (*i*) is differentiable with respect to the operator norm topology, and $U'(t) = iAU(t)$ for all $t \in \mathbb{R}$.

(*iii*) For $\lambda \notin \sigma(A)$ we have $\|(A-\lambda\mathbb{I})^{-1}\| = \operatorname{dist}(\lambda, \sigma(A))^{-1}$.

Problem 43. Let A be the integral operator on $L^2([0,1])$ given by
$$Af(x) = \int_0^1 \min(x,y)\, f(y)\, dy\,.$$

(*i*) Prove that A is bounded and self-adjoint.

(*ii*) Find a measure space (M,μ), an isomorphism $U : L^2([0,1]) \to L^2(M,\mu)$, and a bounded measurable function $F : M \to \mathbb{R}$ such that $UAU^* : L^2(M,\mu) \to L^2(M,\mu)$ is the operator of multiplication by F.

Problem 44 (Cyclic vectors I).

(*i*) An $N \times N$ Hermitian matrix has a cyclic vector iff its eigenvalues are all distinct.

(*ii*) Consider the self-adjoint operators A, B on $L^2([-1,1])$, where A is the multiplication by $x \mapsto x$ and B is the multiplication by $x \mapsto x^2$. Prove:

(a) $f : [-1,1] \to \mathbb{R}, x \mapsto 1$ is a cyclic vector of A.

(b) The characteristic function $\chi_{[0,1]}$ is not a cyclic vector of A.

(c) B does not have any cyclic vectors.