Prof. T. Ø. Sørensen PhD
S. Gottwald

Winter term 2015/16
October 23, 2014

Functional Analysis II
Assignment 2

Problem 5 (Projections I). Let V be a linear space and let P be a *projection* on V, that is, a linear map $P : V \to V$ such that $P^2 = P$. Prove:

(*i*) $R(P) = N(I-P)$.

(*ii*) $V = R(P) \oplus N(P)$, where $\oplus$ denotes the direct sum.

Let $\mathcal{H}$ be a Hilbert space. A projection $P : \mathcal{H} \to \mathcal{H}$ is called *orthogonal* if $\mathrm{R}(P) \perp \mathrm{N}(P)$.

(*iii*) Prove that a projection $P : \mathcal{H} \to \mathcal{H}$ is orthogonal iff $P \in \mathcal{B}(\mathcal{H})$ and $P^* = P$.

(*iv*) Let A be a linear subspace of $\mathcal{H}$. Show that there exists a unique orthogonal projection $P_A : \mathcal{H} \to \mathcal{H}$ with $R(P_A) = \overline{A}$. [*Hint:* Projection Theorem.]

Let $P : \mathcal{H} \to \mathcal{H}$ be an orthogonal projection.

(*v*) Calculate $\sigma_p(P)$ and $\sigma(P)$.

(*vi*) Find an explicit expression for $R_\lambda(P) = (P - \lambda I)^{-1}$ whenever $\lambda \in \rho(P)$.

Problem 6. For $w \in \ell^\infty(\mathbb{N})$ let $T_w : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be the componentwise multiplication by $w = (w_1, w_2, \dots)$, i.e.

$$T_w x := (w_1 x_1, w_2 x_2, \dots)$$

(*i*) Show that T_w is bounded and calculate its norm.

(*ii*) Find the explicit action of the adjoint T_w^*.

(*iii*) Find the subsets of w's in $\ell^\infty(\mathbb{N})$ for which $T_w^* T_w = T_w T_w^*$, for which $T_w = T_w^*$, and for which T_w is compact.

(*iv*) Determine $\sigma_p(T_w)$ and prove that $\overline{\sigma_p(T_w)} = \sigma(T_w)$.

Problem 7. Let X be a Banach space and let $T \in \mathcal{B}(X)$ be bijective. Prove:

(*i*) $\sigma(T^{-1}) = \frac{1}{\sigma(T)} := \{\lambda^{-1} \in \mathbb{C} \mid \lambda \in \sigma(T)\}$.

(*ii*) If $Tx = \lambda x$ for some $\lambda \neq 0$ and $x \in X$, then $T^{-1}x = \lambda^{-1}x$.

Problem 8. Let X be a Banach space, let $T \in \mathcal{B}(X)$, let $\rho(T)$ be the resolvent set of T and for $\lambda \in \rho(T)$ let $R_\lambda(T) = (T - \lambda I)^{-1}$ be the resolvent of T at λ. Prove the following:

(*i*) $R_\lambda(T) - R_\mu(T) = (\lambda - \mu)\, R_\lambda(T)\, R_\mu(T)$ for all $\lambda, \mu \in \rho(T)$.

(*ii*) $R_\lambda(T) - R_\lambda(S) = R_\lambda(T)\,(S - T)\, R_\lambda(S)$ for all $S \in \mathcal{B}(X)$ and $\lambda \in \rho(T) \cap \rho(S)$.

(*iii*) If $\lambda \in \mathbb{C}$ is such that $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$ for some $\lambda_0 \in \rho(T)$, then $\lambda \in \rho(T)$ and

$$R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^{n+1} \,.$$

(*iv*) $R_\lambda(T) = -\sum_{n=0}^{\infty} \lambda^{-1-n}\, T^n$ for $|\lambda| > \|T\|$.

(*v*) $\|R_\lambda(T)\| \geqslant (\mathrm{dist}(\lambda, \sigma(T)))^{-1}$ for all $\lambda \in \rho(T)$.

(*vi*) The map $\rho(T) \to \mathcal{B}(X), \lambda \mapsto R_\lambda(T)$ is continuous.

(*vii*) The map in (*vi*) has a derivative, in the sense that

$$\frac{d}{d\lambda} R_\lambda(T) \;:=\; \lim_{h \to 0} \frac{1}{h}\big(R_{\lambda+h}(T) - R_\lambda(T)\big)$$

exists in $\mathcal{B}(X)$. In fact, $\frac{d}{d\lambda} R_\lambda(T) = R_\lambda(T)^2$.