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Functional Analysis II
Assignment 1

Problem 1 (Examples of compact and non-compact operators). Let X and Y be normed spaces. Decide which of the following operators are compact:

(*i*) $T : C[0,1] \to C[0,1],\ Tf(x) = f(0) + xf(1)$.

(*ii*) $id : X \to X$, $x \mapsto x$.

(*iii*) $F \in \mathcal{B}(X,Y)$ with $\dim \operatorname{Ran}(F) < \infty$ (*finite-rank* operator).

Problem 2 (Some properties of compact operators). Let X, Y and Z be Banach spaces. Prove the following statements:

(*i*) $\mathcal{K}(X,Y)$ is a closed subspace of $\mathcal{B}(X,Y)$.

(*ii*) For $A \in \mathcal{B}(X,Y)$ and $B \in \mathcal{B}(Y,Z)$, we have $BA \in \mathcal{K}(X,Z)$ if A or B is compact.

(*iii*) If $\dim X = \infty$ and $T \in \mathcal{K}(X)$, then $0 \in \sigma(T)$.

(*iv*) If $T \in \mathcal{K}(X,Y)$, then $\operatorname{Ran}(T)$ is closed if and only if T is a finite-rank operator.

Problem 3. Let $\mathcal{H}$ be a separable Hilbert space, let $\{\varphi_n\}_n$ be an orthonormal basis of $\mathcal{H}$, and let P_N be the orthogonal projection onto the span of $\{\varphi_1, \ldots, \varphi_N\}$. Prove for $T \in \mathcal{K}(\mathcal{H})$ that

$$T \circ P_N \xrightarrow{\|\cdot\|} T\ , \quad \text{as } N \to \infty.$$

[*Remark:* This shows that *any compact operator on $\mathcal{H}$ is the limit of a sequence of finite-rank operators*. Although this statement can be extended to compact operators on non-separable Hilbert spaces, it cannot be generalized to arbitrary Banach spaces.]

Problem 4 (Multiplication operators). For $1 \leqslant p < \infty$ and $h \in L^\infty[0,1]$ let

$$M_{p,h} : L^p[0,1] \to L^p[0,1], \quad M_{p,h}f(x) := h(x)f(x)$$

denote the operator of multiplication by h.

(*i*) Find the adjoint $M'_{p,h}$ of $M_{p,h}$ as an operator on $L^q[0,1]$, where $1/p + 1/q = 1$.

(*ii*) Prove that $M_{p,h}$ is compact if and only if $h = 0$.

[*Hint:* Find a closed subspace $V \subset L^p[0,1]$ such that $M_{p,h}\big|_V : V \to V$ is surjective.]