Prof. M. Fraas, PhD
A. Groh, S. Gottwald

Summer term 2015
June 15, 2015

# Functional Analysis
# Exercise Sheet 9

## *Weak convergence and topologies*

- *First version deadline:* **June 22** (13:30). *Final hand in deadline:* **July 6** (13:30)

**Exercise 1** (7 points). *Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space, and let $S_1$ be the unit sphere and $B_1$ the closed unit ball in $\mathcal{H}$.*

(i) *What is the norm closure of $S_1$?*

(ii) *Prove that to each $x \in B_1$ there exists a sequence $(x_n)_n \subset S_1$ such that $x_n \xrightarrow{w} x$.*

(iii) *Find the weak sequential closure of $S_1$, i.e. the set of all weak limits of weakly convergent sequences in $S_1$.*

(iv) *Show that a sequence $(x_n)_n \subset \mathcal{H}$ converges strongly to some $x \in \mathcal{H}$ if and only if it converges weakly to $x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.*

**Exercise 2** (8 points). *Let $(e_n)_{n\in\mathbb{N}} \subset c_0$ be such that the kth entry of $e_n$ is equal to $\delta_{nk}$. Prove:*

(i) $(f(e_n))_{n\in\mathbb{N}} \in \ell^1$ *for all* $f \in (c_0)^*$.

(ii) $c_0^* \cong \ell^1$, *i.e.* $c_0^*$ *and* $\ell^1$ *are isometrically isomorphic.*

(iii) *For* $1 < p \leqslant \infty$, $(e_n)_{n\in\mathbb{N}}$ *converges weakly in* $\ell^p$.

**Exercise 3** (5 points). *Consider the sequence $(x^{(n)})_n \subset \ell^\infty \cong (\ell^1)^*$ given by $x_k^{(n)} = 1$ for all $k \leqslant n$ and $x_k^{(n)} = 0$ otherwise. Prove that*

$$x^{(n)} \xrightarrow{w^*} (1, 1, \dots)$$

*but $(x^{(n)})_n$ does not converge weakly.*

**Exercise 4** (5 points). *(i) Show that if* $(f_n)_n \subset C([0,1])$ *converges weakly to some* $f \in C([0,1])$*, then* $f_n(y) \to f(y)$ *for all* $y \in [0,1]$.

*(ii) Find a sequence* $(f_n)_n \subset C([0,1])$ *such that* $f_n(y) \to f(y)$ *for all* $y \in [0,1]$ *but* $(f_n)_n$ *does not converge weakly to* $f$.

**Exercise 5** (5 points). *Decide which of the following operators* $L : X \to X$ *are compact and compute their norms.*

*(i)* $X = L^2([0,2\pi])$, $Lf(x) := \int_0^{2\pi} \sin(x-y)\, f(y)\, dy$.

*(ii)* $X = C([0,1])$, $Lf(x) := \int_0^x f(y)\, dy$.

*(iii)* $X = \ell^2(\mathbb{N})$, $(Lx)_j := j^{-2} x_j$.

*For general informations please visit* http://www.math.lmu.de/~gottwald/15FA/