LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN
MATHEMATISCHES INSTITUT

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Functional Analysis
Exercise Sheet 6

Bounded linear maps

- *First version deadline:* **June 1** (13:30). *Final hand in deadline:* **June 15** (13:30)

Exercise 1 (5 points). (*i*) *Let $\phi \in (\ell^2)^*$ be given by $\phi(x) := \sum_{n=1}^{\infty} \frac{x_n}{n}$ for all $x \in \ell^2$. Compute the norm of ϕ.*

(*ii*) *Let $L : \ell^1 \to \ell^1$ be the so called left-shift operator, given by $L(x_1, x_2, x_3, \dots) := (x_2, x_3, \dots)$. Show that L is a bounded linear map and compute $\|L\|$.*

Exercise 2 (5 points). *Let $p \in (1, \infty)$ and let q be its Hölder conjugate, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For $x \in \ell^p$ and $y \in \ell^q$ let $\langle x, y\rangle := \sum_{j=1}^{\infty} x_j y_j$. Prove that*

$$\|T\| = \sup_{\substack{0 \neq y \in \ell^q \\ 0 \neq x \in \ell^p}} \frac{|\langle y, Tx\rangle|}{\|y\|_q \|x\|_p}$$

for every bounded linear operator $T : \ell^p \to \ell^p$.

Exercise 3 (5 points). *Let $U \subset \mathcal{H}$ be a closed non-trivial subspace of a Hilbert space $\mathcal{H}$. For $x \in \mathcal{H}$ let $Px \in U$ be the closest element in U to x (uniquely given by the Projection Theorem), i.e. $\|Px - x\| = \inf_{y \in U} \|y - x\|$. Show that $P : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator with*

$$\|P\| = 1\,, \quad P^2 = P\,, \quad (y, Px) = (Py, x)\ \forall x, y \in \mathcal{H}.$$

Exercise 4 (5 points). *Let $K \in C([0,1]^2)$ and for $1 \leqslant p \leqslant \infty$ let*

$$T_p : L^p([0,1]) \to L^p([0,1]), (T_p f)(x) := \int_0^1 K(x,y) f(y)\, dy \quad \forall f \in L^p([0,1])\,.$$

Prove:

(*i*) $\|T_1\| \leqslant \sup_{y \in [0,1]} \int_0^1 |K(x,y)|\, dx$.

(*ii*) $\|T_2\| \leqslant \|K\|_{L^2([0,1]^2)}$.

(*iii*) $\|T_\infty\| \leqslant \sup_{x \in [0,1]} \int_0^1 |K(x,y)|\, dy$.

Exercise 5 (5 points). *Let* $(c_{jk})_{j,k\in\mathbb{N}} \subset \mathbb{C}$ *be such that* $a := \sup_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} |c_{jk}| < \infty$ *and* $b := \sup_{j\in\mathbb{N}} \sum_{k\in\mathbb{N}} |c_{jk}| < \infty$. *Prove that* $T : \ell^p \to \ell^p, (Tx)_j := \sum_{k\in\mathbb{N}} c_{jk} x_k$ *defines a bounded linear map with* $\|T\| \leqslant a^{1/p} b^{1/q}$, *where* $p \in (1,\infty)$ *and* q *is its Hölder conjugate.*

Exercise 6 (5 points). *Let* $\tau : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ *be a sesquilinear form on a Hilbert space* $\mathcal{H}$ *with the property that*

$$\exists C > 0 \,:\, |\tau(x,y)| \leqslant C\|x\|\,\|y\| \;\forall x, y \in \mathcal{H}\,.$$

Prove that there exists a unique bounded linear map A *in* $\mathcal{H}$ *such that* $\tau(x,y) = (Ax,y)$ *for all* $x, y \in \mathcal{H}$.

For general informations please visit http://www.math.lmu.de/~gottwald/15FA/