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Functional Analysis
Exercise Sheet 3

More Banach spaces & Compactness

- *First version deadline:* **May 11** (13:30). *Final hand in deadline:* **May 26** (13:30)

In the last lectures we followed the negative result of Riesz on non-compactness of the unit ball in an infinite dimensional Banach space by trying to classify relatively compact subsets of the space of continuous functions. All exercises on this sheet are directly or indirectly connected to this problem. Exercise 1 is a warm up exercise, in exercises 2-4 you will show that various sets of continuos functions are (not) relatively compact. Then exercise 5 is an example that relatively compact subsets can be characterized precisely also in a variety of other spaces. In this sheet there is a statement for the Banach space c_0, which consists of the sequences in $\mathbb{R}$ that converge to zero. Exercise 6 is an artificial exercise designed to show how the diagonal argument works.

Exercise 1 (5 points). (*i*) *Let* $||\cdot||$ *be a norm on* $\mathbb{R}^n$. *Characterize relatively compact subsets of* $\mathbb{R}^n$ *in the topology induced by this norm.*

(*ii*) *Find a set of continuous functions on* $[0,1]$ *that is equicontinuous but not bounded.*

In the following, $C^1([0,1])$ denotes the set of continuous functions $f:[0,1]\to\mathbb{C}$ such that $f|_{(0,1)}\in C^1((0,1))$ and $(f|_{(0,1)})'$ has a continuous extension to $[0,1]$.

Exercise 2 (5 points). *Decide which of the following subsets of* $C([0,1])$ *are equicontinuous and prove your claim:*

(*i*) $\mathcal{F}_1 := \{f\in C^1([0,1])\,|\,f(0)=0, \|f'\|_\infty\leqslant 1\}$,

(*ii*) $\mathcal{F}_2 := \{f\in C([0,1])\,|\,f(0)=0, \|f\|_\infty\leqslant 1\}$,

(*iii*) $\mathcal{F}_3 := \{f\in C^1([0,1])\,|\,\|f\|_\infty\leqslant 1, \|f'\|_2\leqslant 1\}$.

Exercise 3 (5 points). *Let* K *be a continuous function on the square* $[0,1]\times[0,1]$. *Prove that the subset of* $C([0,1])$ *given by*

$$\mathcal{F} := \left\{ f\in C([0,1]) \,\middle|\, \exists\, g\in C([0,1]) \text{ with } \|g\|_\infty\leqslant 1 \text{ s.th. } f(x)=\int_0^1 K(x,y)g(y)\mathrm{d}y \right\}$$

is relatively compact in $(C([0,1]),\|\cdot\|_\infty)$.

Exercise 4 (5 points)**.** *For* $f \in C^1([0,1])$ *define* $\|f\| := \|f\|_\infty + \|f'\|_\infty$. *Prove that* $(C^1([0,1]), \|\cdot\|)$ *is a normed space and show that* $B := \{f \in C^1([0,1]) \,|\, \|f\| \leqslant 1\}$ *is relatively compact in* $(C(0,1), \|\cdot\|_\infty)$ *but not compact in* $(C(0,1), \|\cdot\|_\infty)$.

Exercise 5 (5 points)**.** *Prove that a subset* $\mathcal{A}$ *of the Banach space* c_0 *is relatively compact if and only if there exists* $\{x_n\}_{n\in\mathbb{N}} \in c_0$ *such that all sequences* $\{a_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ *are dominated by* $\{x_n\}_{n\in\mathbb{N}}$, *i.e.* $|a_n| \le x_n$ *for all* $n \in \mathbb{N}$.

Unless written otherwise we always consider the Banach space c_0 *equipped with the norm* $||\{a_n\}_{n\in\mathbb{N}}|| = \sup_{n\in\mathbb{N}} |a_n|$.

Exercise 6 (5 points)**.** *Consider a sequence* $\{f_n\}_{n\in\mathbb{N}}$ *of sequences (for each* $n \in \mathbb{N}$ *fixed* $\{f_n(m)\}_{m\in\mathbb{N}}$ *is a sequence) defined by*

$$f_n(m) := \left[\frac{m}{2^{n-1}}\right] \mod 2,$$

where $[x] \in \mathbb{N}$ *is the largest integer that is strictly smaller than* x. *Explicitly,*

$$\begin{aligned} f_1 &= 0\,1\,0\,1\,0\,1\ldots \\ f_2 &= 0\,0\,1\,1\,0\,0\ldots \\ f_3 &= 0\,0\,0\,0\,1\,1\,1\,1\,0\ldots \end{aligned}$$

Let $m_N(j) := 2^N j$ *and show that* $\lim_{j\to\infty} f_n(m_N(j))$ *exists for all* $n \le N$. *Observe that for fixed* j, $\lim_{N\to\infty} m_N(j) = \infty$, *and prove that for the diagonal subsequence given by* $m_{diag}(j) := m_j(j)$ *the limit* $\lim_{j\to\infty} f_n(m_{diag}(j))$ *exists for all* $n \in \mathbb{N}$.

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