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Summer term 2015
Apr 27, 2015

Functional Analysis
Exercise Sheet 2

Normed linear spaces & Banach spaces

- *First version deadline:* **May 4** (13:00). *Final hand in deadline:* **May 18** (13:00)

A Normed linear space and especially a Banach space is a basic unit of analysis in FA. In a finite dimensional space the linear and the topological structures are independent, in the sense that all topologies induced by norms are equivalent. On the other hand, in an infinite dimensional space these structures "interact" with each other and different norms can lead to very different notions of convergence, continuity, etc., as we will explore on this and future exercise sheets.

Exercise 1 (5 points). *Let $S_n := \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ be the unit sphere in the n-dimensional Euclidian space equipped with the norm $||\cdot||_2$. Compute*

$$\max_{x\in S_n} ||x||_1 \quad \text{and} \quad \min_{x\in S_n} ||x||_\infty .$$

Find all points where this max/min *is achieved and compute the distance between neighboring points. Sketch how $x \mapsto ||x||_1$ and $x \mapsto ||x||_\infty$ look on S_n.*

Exercise 2 (5 points). *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real normed spaces, and for $z \in X \times Y$ define $\|z\|' := \|x\|_X + \|y\|_Y$ whenever $z = (x, y)$ with $x \in X$ and $y \in Y$. Prove that*

(i) $(X \times Y, \|\cdot\|')$ *is a normed space.*

(ii) If $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *are Banach spaces then so is* $(X \times Y, \|\cdot\|')$.

Exercise 3 (5 points). *Let c_c denote the space of real (or complex) valued sequences with finite support, i.e. $x \in c_c$ iff $x_j \neq 0$ for at most finitely many j. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are not equivalent on c_c.*

Exercise 4 (5 points). *Let $C([0, 1])$ denote the space of continuous functions on the unit interval, and for $f \in C([0, 1])$ let*

$$||f||_1 := \int_0^1 |f(x)| \mathrm{d}x \, .$$

Prove:

(*i*) $(C([0,1]), \|\cdot\|_1)$ *is a normed space.*

(*ii*) $(C([0,1]), \|\cdot\|_1)$ *is not a Banach space.*

Exercise 5 (5 points)**.** *Let* $(X, \mathcal{T})$ *be a topological space, let* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$*, and let*

$$C_b(X) := \Big\{ f : X \to \mathbb{K} \,|\, f \text{ continuous and } \sup_{x \in X} |f(x)| < \infty \Big\}$$

denote the set of bounded continuous functions on X*. Prove that* $(C_b(X), \|\cdot\|_\infty)$ *is a Banach space, where* $\|f\|_\infty := \sup_{x\in X} |f(x)|$ *for any* $f \in C_b(X)$*.*