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Functional Analysis
Exercise Sheet 1

Infinite systems of linear equations

- *First version deadline:* **April 27** (13:00). *Final hand in deadline:* **May 11** (13:00)

Infinite systems of linear equations arise in many direct attempts to solve differential and integral equations. Many great mathematicians working at the end of 19-th and the beginning of 20-th century studied these systems. Questions about solvability of these systems, the notion of solution, etc are among the roots of functional analysis. Here we explore the problem.

Exercise 1 (5 points). *A system of N linear equations for N unknowns x_m has the form*

$$\sum_{m=1}^{N} a_{nm} x_m = y_n, \qquad n = 1, \ldots, N, \tag{1}$$

where a_{nm} and y_n are given real (or complex) numbers. The left hand side of the equation is a linear mapping $A : \mathbb{R}^N \to \mathbb{R}^N$ of a vector $x = (x_1, \cdots, x_N)$ and we can write the equation in matrix notation as

$$Ax = y. \tag{2}$$

Prove that the following statements are equivalent:

(i) The equation $Ax = 0$ has a unique solution $x = 0$,

(ii) There exists a solution of Eq. (1) for any $y \in \mathbb{R}^N$.

Recall that (i) is true provided $\det A \neq 0$.

Infinite series of equations often occurred through series expansion of an unknown function. Here is one of the first examples. In 1886 G. W. Hill studied certain aspects of the motion of the Moon described by an ordinary differential equation

$$y''(x) = 4(\omega^2 + q(x))y(x), \tag{3}$$

where ω is a positive real number, q is a real periodic function with a Fourier expansion $q(x) = 2\sum_{n=1}^{\infty} t_n \cos(2nx)$, and y is an unknown function to be determined.

Exercise 2 (5 points). *Look for a periodic solution y with a Fourier expansion of the form $y(x) = 1/2y_0 + \sum_{n=1}^{\infty} y_n \cos(2nx)$ and conclude that the coefficients y_n satisfy an infinite system of linear equations,*

$$\left(\delta_{n0} + \frac{t_n}{\omega^2 + n^2}\right) y_0 + \sum_{m=1}^{\infty} \left(\delta_{nm} + \frac{t_{|n-m|} + t_{|n+m|}}{\omega^2 + n^2}\right) y_m = 0, \quad n = 0, \ldots, \infty. \tag{4}$$

We defined $t_0 = 0$. The Kronecker delta function δ_{nm} is equal to 1 if $n = m$ and is zero otherwise.

Assumptions: We are in the 19th century (i.e. do not worry about making a rigorous computation).

Until the beginning of 20-th century mathematicians solved infinite systems of linear equations by first solving a chopped system of first N equations and then by taking N to infinity.

Exercise 3 (5 points)**.** *Show that a matrix $A^{(N)}(\omega)$ corresponding to the first $N+1$ equations of system Eq. (4) is given by*

$$A^{(N)}_{nm}(\omega) := \left(\delta_{n0} + \frac{t_n}{\omega^2+n^2}\right)\delta_{m0} + \left(\delta_{nm} + \frac{t_{|n-m|}+t_{|n+m|}}{\omega^2+n^2}\right)(1-\delta_{m0}),$$

for n, $m = 0, \ldots, N$. Assume that $q(x) = cos(2x)$ (i.e. $t_n = \frac{1}{2}\delta_{n1}$)[1] *and prove that the determinant $\det(A^{(N)}(\omega))$ of the corresponding matrix has a limit as $N \to \infty$. Lets call this limit $f(\omega)$.*

Plot the limit function f (e.g. by using a computer). Zeros of this function correspond to values of the parameter ω for which there exists a nontrivial periodic solution of Equation (3).

The mathematical community also quickly realized that solving infinite systems by finite approximations does not always give the correct result. Here is an example.

Exercise 4 (5 points)**.** *Consider an infinite system of equations $\sum_{n=m}^{\infty} x_n = 1$ for $m = 1, \ldots \infty$, i.e.*

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots &= 1 \\ x_2 + x_3 + \cdots &= 1 \\ x_3 + \cdots &= 1 \\ &\vdots \end{aligned}$$

Find a solution $x^{(N)} \in \mathbb{R}^N$ of the chopped system $\sum_{n=j}^{N} x_n = 1$ for $j = 1, \ldots, N$ for the first N unknowns. Does this lead to a solution $(x_1, x_2, \ldots)$ of the original system by setting $x_j = \lim_{N\to\infty} x_j^{(N)}$ for $j = 1, 2, \ldots$?

Of course finite approximations can work well:

Exercise 5 (5 points)**.** *Find a non-trivial*[2] *infinite system $\sum_{m=1}^{\infty} a_{nm}x_m = y_n$, $n = 1, \ldots, \infty$, for which a solution $x^{(N)}$ of the first N equations approaches a solution $(x_1, x_2, \ldots)$ of the full system, in the sense that $x_j = \lim_{N\to\infty} x_j^{(N)}$ for $j = 1, 2, \ldots$.*

Fourier used infinite systems of linear equations to compute coefficients in his series. He considered the equation

$$1 = \sum_{n=1}^{\infty} x_n \cos((2n-1)t), \quad t \in (-\pi/2, \pi/2).$$

[1] Eq. (3) is then called Mathieu equation, this exercise is connected to `http://en.wikipedia.org/wiki/Mathieu_function#Periodic_solutions`.

[2] In this case, *non-trivial* means that the corresponding infinite matrix is not the zero matrix given by $a_{nm} = 0$ for all $n, m \in \mathbb{N}$.

By subsequent differentiation of the equation and evaluating at $t = 0$ he obtained the system of equations

$$
\begin{aligned}
x_1 + x_2 + x_3 + x_4 + \ldots &= 1 \\
x_1 + 3^2 x_2 + 5^2 x_3 + 7^2 x_4 + \ldots &= 0 \\
x_1 + 3^4 x_2 + 5^4 x_3 + 7^4 x_4 + \ldots &= 0 \\
\vdots \qquad\qquad \ldots &= 0
\end{aligned}
$$

He solved the chopped system of the first N equations for N unknowns and then obtained x_n by taking the limit $N \to \infty$.

Exercise 6 (5 points)**.** *Try to obtain coefficients* $x_1, x_2, \ldots$ *by the above method. Does the method make any sense?*

The computation is quite tedious. Instead of doing it by yourself you might want to check Fourier's original solution in his treatise "The Analytical Theory of Heat" Chapter 3, Section II.

For general informations please visit http://www.math.lmu.de/~gottwald/15FA/