LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN

MATHEMATISCHES INSTITUT

Prof. T. Ø. Sørensen PhD
S. Gottwald

Winter term 2014/15
Jan 08, 2014

Functional Analysis II

Assignment 12

Problem 45 (Cyclic Vectors II). Consider the self-adjoint operators A, B in $L^2([-1,1])$ discussed in Problem 44, i.e. $Af(x) = xf(x)$ and $Bf(x) = x^2 f(x)$. Prove:

(*i*) $L^2([-1,1]) \cong \mathcal{H}_1 \oplus \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces on which B has a cyclic vector.

(*ii*) $f \in L^2([-1,1])$ is a cyclic vector for A iff $f(x) \neq 0$ almost everywhere.

(*iii*) An $N \times N$ hermitian matrix has a cyclic vector iff its eigenvalues are all distinct.

Problem 46 (Unbounded multiplication operators). Let X be a metric space and μ a positive measure on the Borel σ-algebra of X such that $\mu(\Lambda) < \infty$ for any bounded Borel set $\Lambda \subset X$. For a (possibly unbounded) measurable function $\phi : X \to \mathbb{C}$ consider the linear map M_ϕ in $L^2(X,\mu)$ defined by

$$\begin{aligned} \mathcal{D}(M_\phi) &:= \left\{ f \in L^2(X,\mu) \;\middle|\; \phi f \in L^2(X,\mu) \right\} \\ M_\phi f &:= \phi f \,. \end{aligned}$$

Prove:

(*i*) $\mathcal{D}(M_\phi)$ is dense in $L^2(X,\mu)$.

(*ii*) $M_\phi^* = M_{\overline{\phi}}$.

(*iii*) $\sigma(M_\phi) = \operatorname{essran}\phi = \{\lambda \in \mathbb{C} \,|\, \forall \varepsilon > 0 : \mu(\{x \in X \,|\, |\lambda - \phi(x)| < \varepsilon\}) > 0\}$.

(*iv*) λ is an eigenvalue of M_ϕ iff $\mu(\phi^{-1}(\{\lambda\})) > 0$.

(*v*) Let $X = \mathbb{R}$, let μ be the Lebesgue measure on $\mathbb{R}$, and let $\phi(x) := x \;\; \forall x \in \mathbb{R}$. Then the position operator $q := M_\phi$ is self-adjoint, has no eigenvalues, and $\sigma(q) = \mathbb{R}$.

Problem 47 (Properties of the adjoint). Let A and B be densely defined operators on a Hilbert space $\mathcal{H}$. Prove:

(*i*) $(\alpha A)^* = \overline{\alpha} A^* \quad \forall \alpha \in \mathbb{C}$.

(*ii*) If $\mathcal{D}(A) \cap \mathcal{D}(B)$ and $\mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ are dense in $\mathcal{H}$ then $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$, $\mathcal{D}(A^*+B^*) = \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$, and $(A+B)^* \supset A^*+B^*$.

(*iii*) If $\mathcal{D}(AB)$ is dense, then $(AB)^* \supset B^* A^*$.

(*iv*) If $A \subset B$ then $A^* \supset B^*$.

(v) If A is self-adjoint then A has no symmetric extensions.

(vi) $N(A^*) = R(A)^\perp$.

Prolbem 48 (von Neumann's Theorem).

(i) Let A be a symmetric operator on a Hilbert space $\mathcal{H}$ and assume there exists a map $C : \mathcal{H} \to \mathcal{H}$ with the following properties:

(a) C is anti-linear (i.e. $C(\alpha x + y) = \overline{\alpha x} C(x) + C(y)$).

(b) C is norm-preserving.

(c) $C^2 = \mathbb{I}$.

(d) $\mathcal{D}(A)$ is invariant under C.

(e) $AC = CA$ on $\mathcal{D}(A)$.

[*Remark:* A map satisfying (a) − (c) is called *conjugation.*]

Prove that A has self-adjoint extensions.

(ii) Consider the operator H in $L^2(\mathbb{R}^d)$ given by

$$\begin{aligned} \mathcal{D}(H) &= C_0^\infty(\mathbb{R}^d) \\ (H\psi)(x) &= -\Delta\psi(x) + V(x)\psi(x) \qquad \text{for a.e. } x \in \mathbb{R}^d, \end{aligned}$$

where $\Delta = \sum_{j=1}^d \partial_j^2$ and $V \in L^2_{loc}(\mathbb{R}^d)$ is real-valued. Show that H is symmetric and has at least one self-adjoint extension.