Mathematisches Institut
der Universität München
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Summer term 2014
Sheet 12
25.06.2014

# Functional Analysis

**E46** [8 points].

(*i*) Let $p \in [1, \infty]$, $d \geqslant 1$ and $m : \mathbb{R}^d \to \mathbb{K}$ be measurable. Let $A$ be the multiplication operator given by

$$A : \mathrm{dom}(A) \to L^p(\mathbb{R}^d), \quad Af(x) := m(x)f(x)\,,$$

where $\mathrm{dom}(A) := \{f \in L^p(\mathbb{R}^d) \,|\, mf \in L^p(\mathbb{R}^d)\}$. Prove that $A$ is closed.

[*Hint:* You may use the following result from measure theory without proof (see e.g. Lemma 2.206 in the Analysis 3 script of Prof. Merkl from 2014): *If $f_n \to f$ in $L^p(X, \mu)$ as $n \to \infty$, then $(f_n)_{n\in\mathbb{N}}$ admits a subsequence that converges pointwise $\mu$-a.e. to $f$.*]

(*ii*) Let $X, Y$ and $Z$ be Banach spaces, let $T : \mathrm{dom}(T) \to Y$ be a closed linear operator with $\mathrm{dom}(T) \subset X$ and let $S \in \mathrm{BL}(Y, Z)$ be invertible. Conclude that $ST$ is closed.

**E47** [4 points]. Prove the following version of the closed graph theorem: *If $X$ and $Y$ are Banach spaces, then a linear operator $T : X \to Y$ is closed if and only if it is continuous.*

**E48** [4 points]. Let $\mathcal{H}$ be a Hilbert space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and let $x \in \mathcal{H}$. Prove that $x_n \to x$ in $\mathcal{H}$, if and only if $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$.

**E49** [8 points]. Let $(X, \|\cdot\|)$ be a normed space with $\dim(X) = \infty$.

(*i*) Prove for $F_1, \ldots, F_n \in X^*$ that $\bigcap_{j=1}^n \ker F_j \neq \{0\}$.

(*ii*) Let $U$ be a neighbourhood of $0 \in X$ with respect to the weak topology. Show that $U$ contains a non-trivial[1] subspace $V$ of $X$.

(*iii*) Prove that the weak topology of $X$ is not first countable.

[*Hint:* By contradiction assume that the weak topology is first countable and use part (*ii*) to construct an unbounded sequence in $X$ that converges weakly to 0.]

*Please hand in your solutions until next* **Wednesday (02.07.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

*For more details please visit* http://www.math.lmu.de/~gottwald/14FA/

[1] I.e. $V \neq \{0\}$.