Mathematisches Institut
der Universität München
Prof. Dr. Peter Müller

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Sheet 9
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# Functional Analysis

*The following problem set was the mid-term exam of last years course. Hence, if you want to experience a **trial exam**, try to solve this exercise sheet in **90 minutes** without using any of your notes.*

**E33** [4 points]. For $j,k \in \mathbb{N}$ let $t_{j,k} \in \mathbb{C}$ be such that $C := \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|t_{jk}|^2 < \infty$. For $x \in \ell^2$ and $j \in \mathbb{N}$ define

$$(Tx)_j := \sum_{k=1}^{\infty} t_{jk}x_k \qquad (*)$$

Prove the following statements:

(*i*) The series $(*)$ is absolutely convergent for each $j \in \mathbb{N}$, and $Tx \in \ell^2$ for all $x \in \ell^2$.

(*ii*) $(*)$ defines a bounded linear operator $T : \ell^2 \to \ell^2$.

**E34** [4 points]. Let $X$ be a Banach space and $A \in \mathrm{BL}(X)$. Show that $\lim_{n\to\infty}\sum_{j=0}^{n}\frac{1}{j!}A^j$ exists in $\mathrm{BL}(X)$.

**E35** [4 points]. Let $X$ be a compact metric space, and let $f, f_n \in C(X,\mathbb{R})$ for all $n \in \mathbb{N}$ be such that $f_n(x) \to f(x)$ for all $x \in X$ as $n \to \infty$ and $f_n(x) \leqslant f_{n+1}(x)$ for all $x \in X$ and all $n \in \mathbb{N}$. Prove that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to $f$ as $n \to \infty$.

[*Hint:* Given $\varepsilon > 0$, show that the sets given by $A_n := \{x \in X : f(x) - f_n(x) < \varepsilon\}$ for $n \in \mathbb{N}$ form an open cover $\{A_n\}_{n\in\mathbb{N}}$ of $X$ satisfying $A_n \subset A_m$ for $n \leqslant m$.]

**E36** [5 points]. We call a point $x \in X$ in a topological space $(X,\mathcal{T})$ *isolated*, if $\{x\} \in \mathcal{T}$. Now, let $X$ be an infinite set and let $(X,d)$ be a complete metric space without isolated points. Prove that $X$ is not countable. [*Hint:* Baire's theorem.]

**E37** [7 points]. Let $M := \{x \in c_c : \sum_{n=1}^{\infty} x_n = 0\}$, where $c_c$ is the space of finitely supported sequences, i.e. $c_c := \{x \in \ell^\infty : x_n \neq 0 \text{ for at most finitely many } n \in \mathbb{N}\}$.

(*i*) Show that $M$ is dense in $\ell^2$.

(*ii*) Compute the orthogonal complement $M^\perp$ of $M$ in $\ell^2$.

(*iii*) Show that $M$ is not dense in $\ell^1$. [*Hint:* One approach is to find a continuous functional $f \in (\ell^1)^*$ such that $M \subset \ker f$.]

*Please hand in your solutions until next* **Wednesday (11.06.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

*For more details please visit* http://www.math.lmu.de/~gottwald/14FA/