Mathematisches Institut
der Universität München
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Summer term 2014
Sheet 7
21.05.2014

Functional Analysis

E25 [5 points]. Let $k \in C([0,1]^2)$ and for $f \in C([0,1])$ and $x \in [0,1]$ define

$$(Tf)(x) := \int_0^x k(x,y)\, f(y)\, dy\,.$$

Prove the following:

(*i*) This defines a bounded linear map $T : (C([0,1]), \|\cdot\|_\infty) \to (C([0,1]), \|\cdot\|_\infty)$.

(*ii*) For every sequence $(f_n)_{n\in\mathbb{N}}$ in $C([0,1])$ with $\|f_n\|_\infty \leqslant 1\, \forall n \in \mathbb{N}$, the sequence of images $(Tf_n)_{n\in\mathbb{N}}$ has a convergent subsequence.

E26 [5 points]. Let $f \in (\ell^\infty)^*$, let $\{e_n\}_{n\in\mathbb{N}}$ be given by $(e_n)_k := \delta_{nk}$ for $k \in \mathbb{N}$ and let $J : c_0^* \to (\ell^\infty)^*$ be the linear isometry from part (*iii*) of E24, i.e. $Jg(x) := \sum_n g(e_n)x_n$ for all $x \in \ell^\infty$ and $g \in c_0^*$.

(*i*) Show that $(f(e_n))_{n\in\mathbb{N}} \in \ell^1$.

(*ii*) Prove that f has a unique representation $f = f_1 + f_2$, where $f_1 \in J(c_0^*)$ and $f_2|_{c_0} = 0$.

E27 [6 points].

(*i*) Let $\mathcal{H}$ be an inner product space and let $L, M \subset \mathcal{H}$ be non-empty. Prove:
   (1) $M^\perp$ is a closed subspace of $\mathcal{H}$.
   (2) $L \subset M$ implies $L^\perp \supset M^\perp$.
   (3) $M \cap M^\perp \subset \{0\}$, $M \subset (M^\perp)^\perp$ and $M^\perp = ((M^\perp)^\perp)^\perp$.
   (4) $M^\perp = \big(\overline{\operatorname{span} M}\big)^\perp$.

(*ii*) Let $\mathcal{H} = C([-1,1])$ be equipped with $\langle f, g\rangle := \int_{-1}^1 \overline{f(x)} g(x)\, dx$. Compute the orthogonal complement of the set $M := \{f \in \mathcal{H} \mid f(x) = f(-x)\, \forall x \in [0,1]\}$.

E28 [8 points]. Let $C([a,b])$ be equipped with $\|\cdot\|_\infty$. Let $a, b, t_0 \in \mathbb{R}$ with $a < t_0 < b$ and let $h \in C([a,b])$.

(*i*) Define $A : C([a,b]) \to C([a,b])$ by $(Af)(t) := \int_{t_0}^t h(s)f(s)\, ds$ for all $t \in [a,b]$. Show that for each $n \in \mathbb{N}$

$$\|A^n\| \leqslant \frac{|b-a|^n}{n!}\, \|h\|_\infty^n\,.$$

(*ii*) Show that for $G \in C([a,b])$ there is a unique $f \in C([a,b])$ such that $(I-A)f = G$.

(*iii*) Conclude that for $x_0 \in \mathbb{R}$ and $g \in C([a,b])$ there exists a unique $f \in C^1([a,b])$ such that for all $t \in [a,b]$

$$f'(t) - h(t)f(t) = g(t)\,, \qquad f(t_0) = x_0\,.$$

Please hand in your solutions until next **Wednesday (28.05.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/