Mathematisches Institut
der Universität München
Prof. Dr. Peter Müller

Summer term 2014
Sheet 4
30.04.2014

Functional Analysis

E13 [8 points]. Let

$$C^1([0,1]) := \left\{ f \in C([0,1]) \;\middle|\; \begin{array}{l} f|_{(0,1)} \in C^1((0,1)) \text{ and } f|'_{(0,1)} \text{ has} \\ \text{a continuous extension to } [0,1] \end{array} \right\}$$

and for $f \in C^1([0,1])$ define $\|f\| := \|f\|_\infty + \|f'\|_\infty$.

(*i*) Prove that $(C^1([0,1]), \|\cdot\|)$ is a normed space.

(*ii*) Show that $\bar{B}_1(0) = \{f \in C^1([0,1]) : \|f\| \leqslant 1\}$ is relatively compact in $(C([0,1]), \|\cdot\|_\infty)$ but not compact in $(C([0,1]), \|\cdot\|_\infty)$.

E14 [4 points]. Let $(X, \mathcal{T})$ be a topological space. Prove the remaining part of Lemma 1.58, i.e. show the implications $(iii) \Rightarrow (iv) \Rightarrow (i)$ of the following statements:

(*i*) If $A_n \subset X$ is open and dense for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} A_n$ is dense in X.

(*iii*) If $A \subset X$ is open and non-empty, then A is non-meagre.

(*iv*) If $A \subset X$ is meagre, then $X \backslash A$ is dense in X.

E15 [4 points]. Let $L \subset X$ be a finite-dimensional subspace of a normed space $(X, \|\cdot\|)$. Prove that L is closed and complete.

E16 [8 points]. For $n \in \mathbb{N}$ let $A_n \subset C([0,1])$ be defined by

$$A_n := \big\{ f \in C([0,1]) \,\big|\, \exists x \in [0,1]\, \forall y \in [0,1] : |f(x) - f(y)| \leqslant n|x-y| \big\}\,.$$

Show that A_n is closed and nowhere dense in $(C([0,1]), \|\cdot\|_\infty)$. Conclude that the set

$$N := \big\{ f \in C([0,1]) \,\big|\, f \text{ is nowhere differentiable} \big\}$$

is dense in $(C([0,1]), \|\cdot\|_\infty)$.

[*Hint:* In order to prove that A_n is nowhere dense, it will be helpful to consider a piecewise linear function $s : [0,1] \to [0, C]$ with slope $\pm(n+D)$ for some constants $C, D > 0$.]

Please hand in your solutions until next **Wednesday (07.05.2014)** *before* **12:00** *in the designated box on the first floor. Don't forget to put your name and the letter of your exercise group on all of the sheets you submit.*

For more details please visit http://www.math.lmu.de/~gottwald/14FA/