ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 8

17.12.2019

Emanuela Giacomelli – emanuela.giacomelli@math.lmu.de

Exercise 1. [20 points]

Suppose that $f_1, f_2, f_3, \cdots$ is a sequence of functions in $H^1(\mathbb{R}^n)$ such that $f_j \rightharpoonup f$ and $(\nabla f_j)_i \rightharpoonup g_i$ for $i = 1, 2, \cdots, n$ weakly in $L^2(\mathbb{R}^n)$. Prove that f is in $H^1(\mathbb{R}^n)$ and that $g_i = (\nabla f)_i$.

Exercise 2. [20 points]

Let f be in $L^p(\mathbb{R}^n)$ with $1 \leq p \leq \infty$. Define

$$g_t := (e^{t\Delta} f)(x) = \int_{\mathbb{R}^n} e^{t\Delta}(x, y) f(y)\, dy,$$

where $e^{t\Delta}(x, y)$ is the heat kernel on $\mathbb{R}^n \times \mathbb{R}^n$. Prove that

1. $\Delta g_t = \frac{d}{dt} g_t$
2. $\lim_{t \downarrow 0} g_t = f$ (as a strong limit)