ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 4

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Emanuela Giacomelli – emanuela.giacomelli@math.lmu.de

Exercise 1. [10 points]

1. Let $p > 1$, $1/p + 1/p' = 1$ and let A denotes an arbitrary measurable set of finite measure. Prove that

$$\|f\|_{p,\omega} := \sup_A |A|^{-1/p'} \int_A |f(x)|\, dx,$$

is a norm in the space

$$L^p_\omega(\mathbb{R}^n) := \left\{ \text{f mesurable} \;\Big|\; \sup_{\alpha>0} \alpha |\{x \,:\, |f(x)| > \alpha\}|^{1/p} < \infty \right\}.$$

2. Define

$$\langle f \rangle_{p,w} := \sup_{\alpha>0} \alpha |\{x \,:\, |f(x)| > \alpha\}|^{1/p}. \tag{1}$$

Prove that for $p > 1$, there exist two constants C_1 and C_2, independent of f such that

$$C_1 \langle f \rangle_{p,\omega} \le \|f\|_{p,\omega} \le C_1 \langle f \rangle_{p,\omega}. \tag{2}$$

Exercise 2. [10 points]

Let $f \in L^p(\mathbb{R}^n)$, $h \in L^r(\mathbb{R}^n)$, $g \in L^q_\omega(\mathbb{R}^n)$, with $\infty > p, q, r > 1$ and $1/p + 1/q + 1/r = 2$, prove that

$$\left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(x-y) h(y)\, dx dy \right| \le C(p,q,r,n) \|f\|_p \|g\|_{q,\omega} \|h\|_r, \tag{3}$$

for some constant $C(p, q, r, n)$.

Exercise 3. [10 points]

Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q_\omega(\mathbb{R}^n)$ with $\infty > p, q > 1$. Prove that

$$\|g * f\|_r \le \frac{1}{q'} \left(\frac{n}{|\mathbb{S}^{n-1}|} \right)^{1/q} C(n, n/q, p) \|g\|_{q,\omega} \|f\|_p, \tag{4}$$

with $1/p + 1/q = 1 + 1/r$.

Exercise 4. [10 points]

Let $0 < \lambda < n$ and let $f \in L^{2n/(2n-\lambda)}(\mathbb{R}^n)$, with $f \neq 0$. Prove that that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{f}(x) |x-y|^{-\lambda} f(y)\, dx dy > 0. \tag{5}$$