ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 10

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Exercise 1. [10 points]

Let $\vartheta : \mathbb{R} \to [0,1]$ be a smooth function on the real line with $\vartheta(t) = 1$ for $t \leq 1$ and $\vartheta(t) = 0$ for $t \geq 2$. For $d \in \mathbb{N}$, $\alpha \in (0,d)$, and $x \in \mathbb{R}^d \setminus \{0\}$, we set $f_\alpha(x) := \vartheta(|x|)|x|^{-\alpha}$. Decide for which $\alpha \in (0,d)$ we have $f_\alpha \in H^1(\mathbb{R}^d)$.
Hint: *Use the Sobolev inequalities.*

Exercise 2. [15 points]

For every $\psi \in H^1(\mathbb{R}^d)$, define:

$$\mathcal{E}[\psi] := \int_{\mathbb{R}^d} dx\, |\nabla\psi(x)|^2 + \int_{\mathbb{R}^d} dx\, V(x)|\psi(x)|^2, \tag{1}$$

$$E_0 := \inf\{\mathcal{E}[\psi] \,|\, \psi \in H^1(\mathbb{R}^d), \|\psi\|_{L^2} = 1\}. \tag{2}$$

1. Let $d \in \mathbb{N}$, $d \geq 3$ and $V \in L^{d/2}(\mathbb{R}^d)$ is real-valued. Prove that E_0 is non-negative, $E_0 \geq 0$, provided $\|V\|_{L^{d/2}}$ is sufficiently small.
2. Let $d = 2$ and let $\varepsilon > 0$ and assume that $V \in L^{1+\epsilon}(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ is non-positive, $V \leq 0$, and does not vanish almost everywhere. Prove that $E_0 < 0$.
   Hint: *Pick some* $\vartheta \in C^\infty(\mathbb{R}, [0,1])$ *such that* $\vartheta = 1$ *on* $(-\infty, 1)$ and $\vartheta = 0$ *on* $[2, \infty]$ *and set* $\chi_R(x) := \vartheta(\frac{1}{R} ln|x|)$ *for* $R \geq 1$ *and* $x \in \mathbb{R}^2$. *Use* χ_R *to define a trial function.*
3. Find sufficient assumptions on V in dimension $d = 1$ to have $E_0 < 0$.

Exercise 3. [15 points]

Let $\mathcal{E}(\rho)$ be the Thomas-Fermi functional,

$$\mathcal{E}_{\mathrm{TF}}(\rho) := \frac{3}{5}\int_{\mathbb{R}^3} dx\, \rho(x)^{5/3} - \int_{\mathbb{R}^3} dx\, \frac{Z}{|x|}\rho(x) + D(\rho,\rho), \tag{3}$$

where $Z > 0$ is a fixed parameter and

$$D(\rho,\rho) := \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} dxdy\, \frac{\rho(x)\rho(y)}{|x-y|}. \tag{4}$$

1. Prove that $\mathcal{E}_{\mathrm{TF}}(\rho)$ is a strictly convex functional on the domain $\mathcal{C}_N$ defined as
   $$\mathcal{C}_N := \left\{\rho \,\middle|\, \rho \geq 0,\, \int_{\mathbb{R}^3} \rho < \infty,\, \rho \in L^{5/3}(\mathbb{R}^3)\right\} \cap \left\{\rho \,\middle|\, \int_{\mathbb{R}^3} \rho = N\right\} \tag{5}$$
   Hint: *Use that* $D(\rho_1,\rho_2) \leq D(\rho_1,\rho_1)^{\frac{1}{2}} D(\rho_2,\rho_2)^{\frac{1}{2}}$.
2. Let
   $$E(N) := \inf_{\rho \in \mathcal{C}_N} \mathcal{E}_{\mathrm{TF}}(\rho), \tag{6}$$
   prove that $E(N)$ is a convex function of N.