ADVANCED ANALYSIS – WiSe 2019/20

Exercise sheet 0

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Exercise 1. [10 points]

Let f_n be an increasing sequence of summable functions on the measure space (Ω, Σ, μ) and let $f(x)$ be defined as

$$f(x) := \lim_{n\to+\infty} f_n(x).$$

Prove that f is measurable and

$$\lim_{n\to+\infty} \int_\Omega f_n(x)\, d\mu(x) = \int_\Omega f(x) d\mu(x).$$

Exercise 2. [10 points]

Find a simple condition on $f_n(x)$ so that

$$\sum_{n=0}^{\infty} \int_\Omega f_n(x)\, \mu(dx) = \int_\Omega \left\{ \sum_{n=0}^{\infty} f_n(x) \right\} \mu(dx).$$

Exercise 3. [10 points]

Let $1 \le p \le 2$ and let $0 < b < a$. Prove that

$$(a+b)^p + (a-b)^p \ge 2a^p + p(p-1)a^{p-2}b^2.$$

Exercise 4. [10 points]

Recall that if f and g are functions in $L^p(\Omega)$ with $1 \le p \le 2$, then

$$\|f+g\|_p^p + \|f-g\|_p^p \ge (\|f\|_p + \|g\|_p)^p + |\|f\|_p - \|g\|_p|^p, \tag{1}$$

$$(\|f+g\|_p + \|f-g\|_p)^p + |\|f+g\|_p - \|f-g\|_p|^p \le 2^p(\|f\|_p^p + \|g\|_p^p). \tag{2}$$

Assume that f and g lie on the unit sphere in L^p, i.e., $\|f\|_p = \|g\|_p = 1$. Assume also that $\|f-g\|_p$ is small. Draw a picture of the situation. Then, using exercise 3, explain why (2) shows that the unit sphere is *uniformly convex*. Explain also why (1) shows that the unit sphere is *uniformly smooth*, i.e., it has no corners.