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LARGE DEVIATIONS AND MAXIMUM ENTROPY PRINCIPLE FOR INTERACTING RANDOM FIELDS ON $\mathbb{Z}^d$

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We present a new approach to the principle of large deviations for the empirical field of a Gibbsian random field on the integer lattice $\mathbb{Z}^d$. This approach has two main features. First, we can replace the traditional weak topology by the finer topology of convergence of cylinder probabilities, and thus obtain estimates which are finer and more widely applicable. Second, we obtain as an immediate consequence a limit theorem for conditional distributions under conditions on the empirical field, the limits being those predicted by the maximum entropy principle. This result implies a general version of the equivalence of Gibbs ensembles, stating that every microcanonical limiting state is a grand canonical equilibrium state. We also prove a converse to the last statement, and discuss some applications.

0. Introduction. As is well known, the study of the asymptotic probabilities of large fluctuations of time averages or space averages away from the mean is based on two fundamental principles: the principle of large deviations, and the maximum entropy principle. The former provides the exact rate of exponential decay of the fluctuation probabilities, whereas the latter predicts the limiting conditional distribution under the condition that the fluctuations are large. It is obvious that these principles are intimately related. In this paper, we investigate these principles in the case of interacting random fields on the integer lattice $\mathbb{Z}^d$.

The setup is the following. First, we let $(E, \mathscr{E})$ be any measurable space. We shall assume throughout that $(E, \mathscr{E})$ is standard Borel, but we shall avoid making any explicit topological assumptions on E. So we do not assume that E is Polish. Next we let $S = \mathbb{Z}^d$ be the d-dimensional integer lattice and $(\Omega, \mathscr{F}) = (E, \mathscr{E})^S$ the associated product space. In lattice models of statistical mechanics, $(E, \mathscr{E})$ is called the state space or single spin space and $(\Omega, \mathscr{F})$ the configuration space. We let $\mathscr{P} = \mathscr{P}(\Omega, \mathscr{F})$ denote the set of all probability measures on $(\Omega, \mathscr{F})$ and we write $\mathscr{P}_\Theta$ for the set of all $\mu \in \mathscr{P}$ which are invariant under the shift-group $\Theta = (\theta_i)_{i \in S}$ acting on Ω via

$$(0.1) \qquad (\theta_i \omega)_j = \omega_{j+i}, \qquad i, j \in S,\ \omega = (\omega_k)_{k \in S} \in \Omega.$$

An element of $\mathscr{P}$ is often called a random field on $\mathbb{Z}^d$. We shall be concerned

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with spatial averages

$$|\Lambda|^{-1} \sum_{i \in \Lambda} f \circ \theta_i \tag{0.2}$$

of bounded local functions $f: \Omega \to \mathbb{R}$ over cubes $\Lambda \subset S$ in the limit as $|\Lambda| \to \infty$. Here, by a cube (with side length p) we mean any set Λ of the form

$$\Lambda = S \cap \prod_{k=1}^{d} [m_k, m_k + p - 1] \tag{0.3}$$

with $m = (m_1, \dots, m_d) \in S$ and $p \in \mathbb{N}$, and the notation $|\Lambda| \to \infty$ means that Λ runs through an arbitrary sequence of cubes whose cardinalities tend to infinity. Moreover, the term "local function" is used synonymously with "cylinder function." That is, a function f on Ω is called local if, for some finite $\Lambda \subset S$, f is measurable relative to the σ-algebra $\mathscr{F}_\Lambda$ generated by the projection $X_\Lambda: \Omega \to E^\Lambda$, $X_\Lambda(\omega) = \omega_\Lambda = (\omega_i)_{i \in \Lambda}$. We let $\mathscr{L}$ denote the set of all bounded local functions $f: \Omega \to \mathbb{R}$.

As is well known, the collective asymptotic behaviour of all spatial averages (0.2) with $f \in \mathscr{L}$ can be described conveniently by a single quantity, the *periodic empirical field*. For a given cube Λ and a configuration $\omega \in E^\Lambda$, this is defined by

$$\rho_\Lambda^\omega = |\Lambda|^{-1} \sum_{i \in \Lambda} \delta_{\theta_i \omega^{\mathrm{per}}}. \tag{0.4}$$

Here δ_ζ stands for the Dirac measure at ζ, and $\omega^{\mathrm{per}} \in \Omega$ is the periodic continuation of ω. That is, if Λ is given by (0.3) then $(\omega^{\mathrm{per}})_j = \omega_i$ whenever $j \in S$ and $i \in \Lambda$ are such that $j_k \equiv i_k \bmod p$ for all $1 \le k \le d$. The main advantage of this periodization is that $\rho_\Lambda^\omega \in \mathscr{P}_\Theta$ for all $\omega \in E^\Lambda$. As $\omega \to \omega^{\mathrm{per}}$ is measurable, $\rho_\Lambda: \omega \to \rho_\Lambda^\omega$ is a measurable function from E^Λ to $\mathscr{P}_\Theta$, provided we equip $\mathscr{P}_\Theta$ (as we shall always do) with the evaluation σ-algebra generated by the mappings $\nu \to \nu(A)$, $A \in \mathscr{F}$. Occasionally, it will be convenient to identify ρ_Λ with the $\mathscr{F}_\Lambda$-measurable function $\omega \to \rho_\Lambda^{X_\Lambda(\omega)}$ on Ω. Accordingly, for each $f \in \mathscr{L}$ we may think of the function

$$\rho_\Lambda f: \omega \to \rho_\Lambda^\omega(f) = \int f \, d\rho_\Lambda^\omega$$

either as a function on E^Λ or as a function on Ω. As a drawback of the periodization, $\rho_\Lambda f$ is in general different from the average (0.2), but it is well known and easy to see that the difference is negligible in the limit $|\Lambda| \to \infty$.

REMARK 0.1. For each $f \in \mathscr{L}$,

$$\lim_{|\Lambda| \to \infty} \left\| \rho_\Lambda f - |\Lambda|^{-1} \sum_{i \in \Lambda} f \circ \theta_i \right\| = 0. \tag{0.5}$$

Here and below, $\|\cdot\|$ stands for the sup norm.

Combining the above remark with the multidimensional ergodic theorem [which can be found in Section 14.A of Georgii (1988), e.g.], we obtain the following information on the asymptotic behaviour of the periodic empirical field: For each ergodic $\mu \in \mathscr{P}_\Theta$ and $f \in \mathscr{L}$, we have

$$\lim_{|\Lambda| \to \infty} \rho_\Lambda f = \int f \, d\mu \quad \text{in } \mathscr{L}^1(\mu), \tag{0.6}$$

and the convergence holds μ-almost surely whenever Λ runs through an increasing sequence of cubes.

We shall be concerned with large deviations from this ergodic behaviour. We start by recalling some terminology. Let (μ_Λ) be a sequence of probability measures μ_Λ on $(E, \mathscr{E})^\Lambda$ indexed by a sequence of cubes Λ with $|\Lambda| \to \infty$. (μ_Λ) is said to satisfy a level-3 large deviation principle with rate function I: $\mathscr{P}_\Theta \to [0, \infty]$ if, for any measurable $C \subset \mathscr{P}_\Theta$,

$$\limsup_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \mu_\Lambda(\rho_\Lambda \in C) \le -\inf I(\overline{C}) \tag{0.7}$$

and

$$\liminf_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \mu_\Lambda(\rho_\Lambda \in C) \ge -\inf I(C^0), \tag{0.8}$$

where $\overline{C}$, respectively, C^0, is the closure, respectively, the interior, of C relative to a suitable topology on $\mathscr{P}_\Theta$. But which topology is suitable? The traditional approach is to assume that E is Polish so that Ω is also Polish, and to equip $\mathscr{P}_\Theta$ with the topology of weak convergence. However, the ergodic theorem (0.6) suggests that the most natural topology on $\mathscr{P}_\Theta$ in this context is the topology $\tau_\mathscr{L}$ defined below. This topology has two advantages:

(i) $\tau_\mathscr{L}$ does not depend on any topology on E, and

(ii) $\tau_\mathscr{L}$ is finer than the weak topology (relative to any Polish topology on E generating $\mathscr{E}$) and thus brings the right-hand sides of (0.7) and (0.8) closer together. In fact, we define $\tau_\mathscr{L}$ on $\mathscr{P}$ rather than only $\mathscr{P}_\Theta$.

DEFINITION 0.2. The *topology $\tau_\mathscr{L}$ of local convergence* is the smallest topology on $\mathscr{P}$ relative to which all evaluation maps

$$\nu \to \nu(f) = \int f \, d\nu, \qquad f \in \mathscr{L},$$

on $\mathscr{P}$ are continuous.

Clearly, $\tau_\mathscr{L}$ can be characterized as the smallest topology on $\mathscr{P}$ such that $\nu \to \nu(A)$ is continuous for each cylinder event $A \in \mathscr{F}$. Equivalently, a net $(\nu^i)_{i \in D}$ in $\mathscr{P}$ converges to some $\nu \in \mathscr{P}$ relative to $\tau_\mathscr{L}$ if and only if, for all finite $\Delta \subset S$, the marginal distributions $\nu^i_\Delta = \nu^i \circ X_\Delta^{-1}$ on $(E, \mathscr{E})^\Delta$ converge to ν_Δ in the so-called *τ-topology* on $\mathscr{P}(E^\Delta, \mathscr{E}^\Delta)$ which is generated by the mappings $\alpha \to \alpha(g)$, g: $E^\Delta \to \mathbb{R}$ bounded and measurable. In other words, $\tau_\mathscr{L}$ is the natural level-3 analogue of the τ-topology on $\mathscr{P}(E, \mathscr{E})$; the latter is frequently considered in the study of large deviations on level 2 concerning the empirical distribution (3.8) [cf., e.g., Groeneboom, Oosterhoff and Ruymgaart (1979), Csiszár (1984) and Bolthausen (1987)]. In the setting of continuous time processes, the analogue of $\tau_{\mathscr{L}}$ appears in Deuschel and Stroock (1989) under the name "projective limit strong topology."

Throughout this article we shall assume that $\mathscr{P}$ is equipped with the topology $\tau_{\mathscr{L}}$. $\mathscr{P}_\Theta$, as a closed subset of $\mathscr{P}$, will always be equipped with the relative topology. Our main objective is a proof of the large deviation principle (0.7) and (0.8) for sequences (μ_Λ) which can be considered as perturbations of a product measure sequence (λ^Λ) with $\lambda \in \mathscr{P}(E, \mathscr{E})$. As we shall see, this includes many interesting cases with nontrivial interaction between the lattice sites, such as the Gibbsian distributions. Although we are mainly interested in these cases of interacting random fields, we shall first prove (an extended version of) the large deviation principle (0.7) and (0.8) for the special case when $\mu_\Lambda = \lambda^\Lambda$ for some $\lambda \in \mathscr{P}(E, \mathscr{E})$. In this case, the rate function I is nothing other than (minus) the specific entropy. The lower bound (0.8) can be obtained by a well-known and natural argument which is based on the Shannon–McMillan–Breiman theorem. Our approach to the upper bound (0.7) arose from an attempt at lifting the ideas of Csiszár (1984) for level-2 large deviations to level 3. Here is an outline of our argument for the upper bound (0.7) in the case $\mu_\Lambda = \lambda^\Lambda$. We start from Csiszár's basic identity

$$|\Lambda|^{-1} \log \lambda^\Lambda(\rho_\Lambda \in C) = -|\Lambda|^{-1} I(\mu_{\Lambda, C}; \lambda^\Lambda). \tag{0.9}$$

Here $I(\cdot\,;\cdot)$ stands for the relative entropy (also called I-divergence or Kullback–Leibler information), and $\mu_{\Lambda,C} = \lambda^\Lambda(\cdot\,|\rho_\Lambda \in C)$ is the associated conditional probability distribution. Then we partition the lattice S into disjoint Λ-blocks, that is, translates of Λ, and consider the periodic measure on $(\Omega, \mathscr{F})$ relative to which the configurations in distinct Λ-blocks are independent with identical distribution $\mu_{\Lambda,C}$. The spatial average $\tilde{\mu}_{\Lambda,C}$ of this periodic measure belongs to $\mathscr{P}_\Theta$, and its negative specific entropy $I(\tilde{\mu}_{\Lambda,C})$ is given by

$$I(\tilde{\mu}_{\Lambda,C}) = |\Lambda|^{-1} I(\mu_{\Lambda,C}; \lambda^\Lambda). \tag{0.10}$$

By definition of $\mu_{\Lambda,C}$ we have $\mu_{\Lambda,C}\rho_\Lambda = \int \mu_{\Lambda,C}(d\omega)\rho_\Lambda^\omega \in \overline{\mathrm{co}}\, C$, the closed convex hull of C. Moreover, the sequences $(\tilde{\mu}_{\Lambda,C})$ and $(\mu_{\Lambda,C}\rho_\Lambda)$ in $\mathscr{P}_\Theta$ have the same set of accumulation points. Since the level sets of I are $\tau_{\mathscr{L}}$-compact, this immediately shows that

$$\liminf_{|\Lambda| \to \infty} I(\tilde{\mu}_{\Lambda,C}) \geq \inf I(\overline{\mathrm{co}}\, C). \tag{0.11}$$

This completes the proof of (0.7) in the case when C is convex. In the general case, we use the fact that I is affine (this is a payoff of working on level 3) to conclude that $\inf I(\overline{\mathrm{co}}\, C) = \inf I(\overline{C})$, whence (0.7) follows from (0.9) to (0.11).

A main advantage of the preceding argument is that it reveals the intimate connection between the principle of large deviations and the maximum entropy principle. Indeed, suppose C is such that the right-hand sides of (0.7) and (0.8) coincide and are finite. It then follows from (0.9) and (0.10) that

$$\varnothing \neq \operatorname*{acc}_{\Lambda \uparrow S} \tilde{\mu}_{\Lambda, C} \subset \{\nu \in \overline{\mathrm{co}}\, C : I(\nu) = \inf I(\overline{C})\},$$

where $\mathrm{acc}_{\Lambda \uparrow S}$ stands for the set of all accumulation points when Λ runs through any sequence of cubes which eventually contain each finite subset of S. But $\mathrm{acc}_{\Lambda \uparrow S}\, \tilde{\mu}_{\Lambda, C} = \mathrm{acc}_{\Lambda \uparrow S}\, \mu_{\Lambda, C}$, whence

$$\varnothing \neq \operatorname*{acc}_{\Lambda \uparrow S} \lambda^{\Lambda}(\cdot \mid \rho_{\Lambda} \in C) \subset \{\nu \in \overline{\mathrm{co}}\, C : I(\nu) = \inf I(\overline{C})\}. \tag{0.12}$$

This is a version of the maximum entropy principle. When applied to specific sets C, it yields a result expressing the equivalence of Gibbs ensembles.

The paper is organized as follows. Section 1 starts with some preliminaries on specific entropy and then contains a statement of main results, namely: a level-3 large deviation principle for independent and, as a consequence, for interacting random fields, and a maximum entropy principle for interacting random fields. In Section 2, these results are applied to obtain a uniform large deviation principle for Gibbs distributions, and a result on the accumulation points of microcanonical Gibbs distributions showing that, in the infinite volume limit, every microcanonical equilibrium state is a grand canonical Gibbs measure. In a sense, the converse is also true. This result can be found in Section 3, together with a discussion of some special cases. Most proofs are deferred to Section 4.

1. Main results.

1.1. *Preliminaries.* Throughout the paper we let λ be a fixed probability measure on a standard Borel space $(E, \mathscr{E})$. The set $\mathscr{P}$ of all probability measures on the product space $(\Omega, \mathscr{F}) = (E, \mathscr{E})^S$ is equipped with the topology $\tau_{\mathscr{L}}$ of local convergence and the evaluation σ-algebra generated by the mappings $\nu \to \nu(f)$, $f\colon \Omega \to \mathbb{R}$ bounded and $\mathscr{F}$-measurable; $\mathscr{P}_{\Theta}$ is endowed with the induced topology and σ-algebra. [Here and below, we write $\nu(f)$ for the integral of f relative to ν.] Also, we let $\mathscr{S}$ denote the set of all finite nonempty subsets of S, and we write $\mathscr{S}_c$ for the set of all cubes.

Next we introduce the specific entropy. For any two probability measures α, β on the same measurable space we let

$$I(\alpha; \beta) = \begin{cases} \alpha(\log f), & \text{if } \alpha \ll \beta \text{ with density } f, \\ \infty, & \text{otherwise}, \end{cases} \tag{1.1}$$

denote the relative entropy of α relative to β. As is well known, the integral above is always well defined and nonnegative. The negative *specific entropy* or *mean entropy* of a measure $\nu \in \mathscr{P}_{\Theta}$ is then defined by

$$I(\nu) = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} I(\nu_{\Lambda}; \lambda^{\Lambda}), \tag{1.2}$$

where $\nu_{\Lambda} = \nu \circ X_{\Lambda}^{-1}$ stands for the marginal distribution of ν on $(E, \mathscr{E})^{\Lambda}$. [Note that the λ-dependence of $I(\nu)$ is suppressed in our notation.] The existence of the limit follows from the multidimensional Shannon–Perez theorem; compare Georgii (1988), Theorem (15.12). The following properties of I are well known:

$$\text{For all } \nu \in \mathscr{P}_\Theta, \quad I(\nu) = \sup_{\Lambda \in \mathscr{S}} |\Lambda|^{-1} I(\nu_\Lambda; \lambda^\Lambda); \tag{1.3}$$

(1.4) I is lower semicontinuous, and its level sets $\{I \le c\}$, $c \ge 0$, are compact and sequentially compact; and

$$I \text{ is an affine function } \mathscr{P}_\Theta. \tag{1.5}$$

For a proof we refer to Georgii (1988), Propositions (15.16), (15.14) and (4.15). Properties (1.4) and (1.5) will be fundamental for our results. [The compactness of the level sets depends on our standard-Borel assumption on $(E, \mathscr{E})$.]

We shall also deal with arbitrary functionals $F\colon \mathscr{P}_\Theta \to]-\infty, \infty]$ and their semicontinuous regularizations defined by

$$F^{\text{usc}}(\nu) = \lim_{U \downarrow \nu} \sup F(U), \qquad F_{\text{lsc}}(\nu) = \lim_{U \downarrow \nu} \inf F(U). \tag{1.6}$$

Here $\nu \in \mathscr{P}_\Theta$, $\lim_{U \downarrow \nu}$ means that the limit is taken along the net of all open neighbourhoods U of ν, and $F(U)$ is the range of F on U. It is easily checked that F^{usc} is the lowest upper semicontinuous majorant and F_{lsc} the largest lower semicontinuous minorant of F.

1.2. *Statement of results.* The first result is a large deviation principle for the periodic empirical fields ρ_Λ defined in (0.4). In fact, in place of the events $\{\rho_\Lambda \in C\}$ which appear in (0.6) and (0.7) we shall more generally look at integrals of functionals $F(\rho_\Lambda)$ of ρ_Λ, or of functions which are asymptotically close to such functions. So Theorem 1.2 should be regarded as a version of the Laplace approximation method. To simplify its statement we introduce the following concept.

DEFINITION 1.1. An *asymptotic empirical functional* $\{(F_\Lambda), F\}$ is a family $(F_\Lambda)_{\Lambda \in \mathscr{S}_c}$ of measurable functions $F_\Lambda\colon E^\Lambda \to]-\infty, \infty]$ indexed by the set $\mathscr{S}_c$ of cubes, together with a functional $F\colon \mathscr{P}_\Theta \to]-\infty, \infty]$ which is bounded from below and such that

$$\lim_{|\Lambda| \to \infty} \big\| |\Lambda|^{-1} F_\Lambda - F(\rho_\Lambda) \big\| = 0. \tag{1.7}$$

Here we use the convention $\infty - \infty = 0$; that is, (1.7) means implicitly that $\{F_\Lambda = \infty\} = \{F(\rho_\Lambda) = \infty\}$ eventually.

THEOREM 1.2. *For any asymptotic empirical functional* $\{(F_\Lambda), F\}$,

$$\limsup_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) \le -\inf[I + F_{\text{lsc}}] \tag{1.8}$$

and

$$\liminf_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) \ge -\inf[I + F^{\text{usc}}]. \tag{1.9}$$

The proof of the upper bound (1.8) will be given in subsection 4.1. The lower bound (1.9) follows by a standard argument set out by Orey (1986), Föllmer (1988) and Föllmer and Orey (1988). This argument is based on the multidimensional version of McMillan's theorem [cf. Nguyen and Zessin (1979) or Tempelman (1984)] and carries over without change to our setting.

It is easily shown by examples that the coincidence of the right-hand sides of (1.8) and (1.9) is by no means necessary for the existence of the limit

$$\lim_{|\Lambda|\to\infty} |\Lambda|^{-1}\log \lambda^{\Lambda}(e^{-F_\Lambda}).$$

Nevertheless, this coincidence is a natural sufficient condition which holds trivially when F is continuous, and this is the advantage of $\tau_{\mathscr{L}}$ over the coarser weak topologies. On the other hand, using (1.5) and Proposition 19.3 of Choquet (1969) one can easily show that the right-hand sides of (1.8) and (1.9) also coincide when F is convex and the right-hand side of (1.9) is finite.

Remark 1.3. Theorem 1.2 implies the large deviation principle (0.7) and (0.8) in the case $\mu_\Lambda = \lambda^\Lambda$. Indeed, let $C \subset \mathscr{P}_\Theta$ be measurable and define an asymptotic empirical functional $\{(F_\Lambda), F\}$ by $F(\nu) = 0$ if $\nu \in C$, $F(\nu) = \infty$ if $\nu \notin C$, and $F_\Lambda = |\Lambda| F(\rho_\Lambda)$. Then $\lambda^\Lambda(e^{-F_\Lambda}) = \lambda^\Lambda(\rho_\Lambda \in C)$, $F_{\mathrm{lsc}} = 0$ on $\overline{C}$ and $= \infty$ on $\mathscr{P}_\Theta \setminus \overline{C}$, $F^{\mathrm{usc}} = 0$ on C^0 and $= \infty$ on $\mathscr{P}_\Theta \setminus C^0$, and thereby $\inf[I + F_{\mathrm{lsc}}] = \inf I(\overline{C})$, $\inf[I + F^{\mathrm{usc}}] = \inf I(C^0)$. Conversely, Theorem 1.2 can be deduced from (0.7) and (0.8) by means of a version of the well-known Laplace–Varadhan method; compare Varadhan (1966, 1988). However, we prefer to give a direct proof of Theorem 1.2 because this does not require any additional effort and will also be useful for obtaining Theorem 1.6.

Remark 1.4. Theorem 1.2 still holds under weaker conditions than (1.7). Namely, it is sufficient to require that $|\Lambda|^{-1}F_\Lambda$ is eventually nearly sandwiched between two values of F in a prescribed neighbourhood of ρ_Λ. To make this precise we let $\mathscr{U}$ denote the system of all sets of the form

$$U = \Big\{(\mu,\nu) \in \mathscr{P}\times\mathscr{P}\colon \max_{1\le i\le n} |\mu(f_i) - \nu(f_i)| < \delta\Big\}$$

with $\delta > 0$, $n \ge 1$ and $f_1,\dots,f_n \in \mathscr{L}$. For $\mu \in \mathscr{P}$ and $U \in \mathscr{U}$ we write $U(\mu)$ for the μ-section of U. Clearly, $\mathscr{U}$ is a uniformity base for $\tau_{\mathscr{L}}$ on $\mathscr{P}$, and the sets $U(\mu)$ with $U \in \mathscr{U}$ form a base of neighbourhoods of μ. We extend I to a functional on $\mathscr{P}$ by setting $I = \infty$ on $\mathscr{P}\setminus\mathscr{P}_\Theta$. We also assume that the functional F is defined on $\mathscr{P}$ (rather than only $\mathscr{P}_\Theta$) and define F^{usc} and F_{lsc} by means of neighbourhoods in $\mathscr{P}$ [rather than $\mathscr{P}_\Theta$, as in (1.6)]. [Note that $\mathscr{P}_\Theta$ is contained in the closure of $\mathscr{P}\setminus\mathscr{P}_\Theta$. The restrictions of the above regularizations to $\mathscr{P}_\Theta$ are thus, in general, different from the functions defined in (1.6). To reobtain the situation of Theorem 1.2 we may set $F = \infty$, respectively, $F = \inf F(\mathscr{P}_\Theta)$, on $\mathscr{P}\setminus\mathscr{P}_\Theta$.]

Suppose now (F_Λ) is a family of measurable functions $F_\Lambda\colon E^\Lambda \to]-\infty, \infty]$ such that, for each $U \in \mathscr{U}$,

$$\liminf_{|\Lambda|\to\infty} \inf_{\omega\in E^\Lambda} \left[|\Lambda|^{-1}F_\Lambda(\omega) - \inf F(U(\rho_\Lambda^\omega))\right] \geq 0. \tag{1.10}$$

Then the inequality (1.8) (with the new meaning of the right-hand side) remains valid. Similarly, if

$$\limsup_{|\Lambda|\to\infty} \sup_{\omega\in E^\Lambda} \left[|\Lambda|^{-1}F_\Lambda(\omega) - \sup F(U(\rho_\Lambda^\omega))\right] \leq 0, \tag{1.11}$$

then (1.9) still holds. The simple proof of these claims is deferred to the end of Subsection 4.1.

The reason for the interest in the above extension of Theorem 1.2 is that the conditions (1.10) and (1.11) are stable under small perturbations of ρ_Λ. For example, consider

$$R_{\Lambda,\eta}^\omega = |\Lambda|^{-1} \sum_{i\in\Lambda} \delta_{\theta_i(\omega\eta_{S\setminus\Lambda})},$$

the *empirical field* of $\omega \in E^\Lambda$ with *boundary condition* $\eta \in \Omega$. Here $\omega\eta_{S\setminus\Lambda} \in \Omega$ is the configuration which equals ω on Λ and η on $S \setminus \Lambda$. Remark 0.1 shows that conditions (1.10) and (1.11) remain unchanged if ρ_Λ^ω is replaced by $R_{\Lambda,\eta}^\omega$ for arbitrary η. In particular, (1.10) and (1.11) hold when $F_\Lambda = |\Lambda|F(R_{\Lambda,\eta})$, and the convergence is uniform in η. The result above thus gives us a large deviation principle for $R_{\Lambda,\eta}$ under λ^Λ which is uniform in η, and therefore also a large deviation principle for the empirical field,

$$R_\Lambda^\omega = |\Lambda|^{-1} \sum_{i\in\Lambda} \delta_{\theta_i\omega}, \qquad \omega \in \Omega,$$

under λ^S. Mutatis mutandis, this remark also applies to the corollaries to Theorem 1.2.

Although Theorem 1.2 is only stated for product measures λ^Λ, it immediately implies a similar result for dependent random fields. This is one of the basic extension principles of large deviation theory; see, for example, Theorem II.7.2. of Ellis (1985). For any asymptotic empirical functional $\{(F_\Lambda), F\}$ and any $\Lambda \in \mathscr{S}_c$ with $\lambda^\Lambda(F_\Lambda < \infty) > 0$ we set

$$\mu_\Lambda^F = \lambda^\Lambda\left(e^{-F_\Lambda}\right)^{-1} e^{-F_\Lambda}\lambda^\Lambda, \tag{1.12}$$

that is, μ_Λ^F is the probability measure on $(E, \mathscr{E})^\Lambda$ with a λ^Λ-density proportional to e^{-F_Λ}. Thus μ_Λ^F has the form of a Gibbs distribution with "Hamiltonian" F_Λ.

COROLLARY 1.5. *Let* $\{(F_\Lambda), F\}$ *and* $\{(G_\Lambda), G\}$ *be two asymptotic empirical functionals. Suppose* F *is continuous and such that* $\inf[I + F] < \infty$. *Then* μ_Λ^F

is eventually well defined, and we have

$$\limsup_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \mu_\Lambda^F(e^{-G_\Lambda}) \le -\inf\left[I^F + G_{\mathrm{lsc}}\right]$$

and

$$\liminf_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \mu_\Lambda^F(e^{-G_\Lambda}) \ge -\inf[I^F + G^{\mathrm{usc}}],$$

where the F-modified entropy functional I^F is defined by $I^F = I + F - \inf[I + F]$. In particular, choosing $\{(G_\Lambda), G\}$ as in Remark 1.3 *it follows that (μ_Λ^F) satisfies a level-*3 *large deviation principle with rate function I^F.*

PROOF. Since F is continuous, $F^{\mathrm{usc}} = F_{\mathrm{lsc}} = F$ and $(F+G)^{\mathrm{usc}} = F + G^{\mathrm{usc}}$, $(F+G)_{\mathrm{lsc}} = F + G_{\mathrm{lsc}}$. Theorem 1.2 thus implies that

$$\lim_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) = -\inf[I+F] > -\infty.$$

In particular, $\lambda^\Lambda(e^{-F_\Lambda}) > 0$ eventually, whence μ_Λ^F is eventually well defined. As $\mu_\Lambda^F(e^{-G_\Lambda}) = \lambda^\Lambda(e^{-F_\Lambda - G_\Lambda})/\lambda^\Lambda(e^{-F_\Lambda})$, the result follows by applying Theorem 1.2 to the asymptotic empirical functional $\{(F_\Lambda + G_\Lambda), F + G\}$. □

As we have pointed out in the introduction, our proof of Theorem 1.2 yields as a by-product a limit theorem for averaged or periodic distributions of Gibbsian type. The limiting distributions are mixtures of random fields which minimize the associated free energy. So this limit theorem can be viewed as a "minimum free energy principle" or, in more conventional terms, a maximum entropy principle. By abuse of notation, we write μ_Λ^F for an arbitrary probability measure on $(\Omega, \mathscr{F})$ whose marginal distribution on $(E, \mathscr{E})^\Lambda$ coincides with the Gibbs distribution (1.12). We also introduce the *averaged Gibbs distribution*

$$\bar{\mu}_\Lambda^F = |\Lambda|^{-1} \sum_{i\in\Lambda} \mu_\Lambda^F \circ \theta_i^{-1}. \tag{1.13}$$

We write $\Lambda \uparrow S$ to indicate that Λ runs through the directed set $\mathscr{S}_c$ of cubes ordered by inclusion. (The more general case when Λ runs through the set of all cubes in a halfspace or an octant only requires trivial modifications.)

THEOREM 1.6. *Let $\{(F_\Lambda), F\}$ be an asymptotic empirical functional satisfying*

$$\inf[I + F_{\mathrm{lsc}}] = \inf[I + F^{\mathrm{usc}}] < \infty. \tag{1.14}$$

Then, in the limit $\Lambda \uparrow S$, the net $(\bar{\mu}_\Lambda^F)$ admits at least one accumulation point $\mu \in \mathscr{P}_\Theta$, and each such μ has a representation $\mu = \int w(d\nu)\nu$ in terms of a Borel probability measure w on the compact set

$$M^F = \{\nu \in \mathscr{P}_\Theta \colon I(\nu) + F_{\mathrm{lsc}}(\nu) = \inf[I + F_{\mathrm{lsc}}]\}. \tag{1.15}$$

In particular, if $M^F = \{\nu^F\}$ then $\lim_{\Lambda\uparrow S} \bar{\mu}_\Lambda^F = \nu^F$. Moreover, if each F_Λ is a function of ρ_Λ, the same conclusions hold when $(\bar{\mu}_\Lambda^F)$ is replaced by (μ_Λ^F).

The proof of Theorem 1.6 is postponed until subsection 4.2. There are simple examples showing that the accumulation points of $(\bar{\mu}_\Lambda^F)$ may form an uncountable subset of $\overline{\text{co}}\,M^F \setminus M^F$, where $\overline{\text{co}}\,M^F$ stands for the closed convex hull of M^F. It is also important to note that, in general, one cannot dispense with the spatial averaging. That is, if the F_Λ's fail to share the Λ-periodicity of the ρ_Λ's then (μ_Λ^F) may admit a non-shift-invariant accumulation point, and the conclusion of Theorem 1.6 fails. This follows from an example of Csiszár, Cover and Choi (1987).

Some applications of Corollary 1.5 and Theorem 1.6 will be discussed in Sections 2 and 3. We shall not, however, treat the straightforward application to the so-called Curie–Weiss or mean-field models for which the rate function I^F and the set M^F can often be determined explicitly. We rather refer to Orey (1988), Ellis (1985), Ellis and Newman (1978), Eisele and Ellis (1988), Ellis and Wang (1990), Messer and Spohn (1982) and Ben Arous and Brunaud (1990).

Finally, we note that the techniques of the present paper can also be applied to systems of marked point particles in $\mathbb{R}^d$. This is done in Georgii and Zessin (1993). This paper also shows that the preceding results still hold when the concept of an asymptotic empirical functional is extended in a way which allows for applications to certain systems with unbounded interaction.

2. Applications to Gibbs measures.

2.1. *Large deviations for Gibbs measures.* We start by recalling the definitions of Gibbs distributions and Gibbs measures relative to an interaction potential. A (shift-invariant, absolutely summable) *potential* is a collection $\Phi = (\Phi_A)_{A \in \mathscr{S}}$ of functions $\Phi_A\colon \Omega \to \mathbb{R}$ satisfying the following:

(P1) For all $A \in \mathscr{S}$, Φ_A is $\mathscr{F}_A$-measurable.
(P2) For all $A \in \mathscr{S}$ and $i \in S$, $\Phi_{A+i} = \Phi_A \circ \theta_i$.
(P3) $\|\Phi\|_0 := \sum_{A \ni 0} \|\Phi_A\| < \infty$.

The *Hamiltonian* for Φ in a region $\Lambda \in \mathscr{S}$ with "boundary condition" $\omega \in \Omega$ is then given by

$$H_\Lambda^{\Phi,\omega}(\zeta) = \sum_{A \cap \Lambda \neq \varnothing} \Phi_A(\zeta\omega_{S \setminus \Lambda}), \qquad \zeta \in E^\Lambda. \tag{2.1}$$

The *Gibbs distribution* $\gamma_\Lambda^{\Phi,\omega}$ in $\Lambda \in \mathscr{S}$ with boundary condition $\omega \in \Omega$ relative to Φ (and the a priori measure λ) is defined as the probability measure on $(E, \mathscr{E})^\Lambda$ with density

$$\exp\left[-H_\Lambda^{\Phi,\omega}\right] / Z_\Lambda^{\Phi,\omega}$$

relative to λ^Λ. The normalizing constant

$$Z_\Lambda^{\Phi,\omega} = \lambda^\Lambda\left(\exp\left[-H_\Lambda^{\Phi,\omega}\right]\right)$$

is known as the partition function. Note that $\gamma_\Lambda^{\Phi,\omega}$ is equivalent to λ^Λ. Clearly, the mapping $\gamma_\Lambda^\Phi\colon (\omega, A) \to \gamma_\Lambda^{\Phi,\omega}(A)$ on $\Omega \times \mathscr{E}^\Lambda$ is a probability kernel from

$(\Omega, \mathscr{F}_{S\setminus\Lambda})$ to $(E, \mathscr{E})^\Lambda$. A probability measure $\mu \in \mathscr{P}$ is called a *Gibbs measure* for Φ (and λ) if, for each $\Lambda \in \mathscr{S}$, μ admits γ_Λ^Φ as a regular conditional marginal distribution on E^Λ relative to $\mathscr{F}_{S\setminus\Lambda}$. We write $\mathscr{G}(\Phi)$ for the set of all Gibbs measures for Φ, and $\mathscr{G}_\Theta(\Phi) = \mathscr{G}(\Phi) \cap \mathscr{P}_\Theta$ for the set of all shift-invariant Gibbs measures. Physically speaking, $\mathscr{G}(\Phi)$ is the set of all equilibrium states for a spin system with interaction Φ. For a detailed account of the theory of Gibbs measures we refer to Georgii (1988).

We now state a uniform large deviation principle for the distribution of the periodic empirical field ρ_Λ under $\gamma_\Lambda^{\Phi,\omega}$. In the case when E is Polish (resp., finite) and $\mathscr{P}_\Theta$ is equipped with the weak topology, this theorem was proved by Comets (1986), Olla (1988) and Föllmer and Orey (1988). In the case of finite range potentials, a completely different proof was recently given by Deuschel, Stroock and Zessin (1991).

THEOREM 2.1. *Let* $\{(G_\Lambda), G\}$ *be an asymptotic empirical functional and* $(\Phi^\Lambda)_{\Lambda\in\mathscr{S}_c}$ *a family of potentials such that* $\lim_{|\Lambda|\to\infty}\|\Phi^\Lambda - \Phi\|_0 = 0$ *for some potential* Φ. *Then*

$$\limsup_{|\Lambda|\to\infty}|\Lambda|^{-1}\log\sup_{\omega\in\Omega}\gamma_\Lambda^{\Phi^\Lambda,\omega}(e^{-G_\Lambda}) \le -\inf\left[I^\Phi + G_{\mathrm{lsc}}\right]$$

and

$$\liminf_{|\Lambda|\to\infty}|\Lambda|^{-1}\log\inf_{\omega\in\Omega}\gamma_\Lambda^{\Phi^\Lambda,\omega}(e^{-G_\Lambda}) \ge -\inf[I^\Phi + G^{\mathrm{usc}}].$$

In the above, $I^\Phi\colon \mathscr{P}_\Theta \to [0,\infty]$ *is defined by*

$$I^\Phi(\nu) = I(\nu) + \langle\nu,\Phi\rangle + P(\Phi), \qquad \nu\in\mathscr{P}_\Theta, \tag{2.2}$$

where

$$\langle\nu,\Phi\rangle = \nu\left(\sum_{A\ni 0}|A|^{-1}\Phi_A\right) = \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\nu_\Lambda\left(H_\Lambda^{\Phi,\omega}\right)$$

and

$$P(\Phi) = -\inf[I + \langle\,\cdot\,,\Phi\rangle] = \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}\log Z_\Lambda^{\Phi,\omega}$$

uniformly in $\omega\in\Omega$. *In particular, for each* $\mu\in\mathscr{G}(\Phi)$ *the sequence* (μ_Λ) *satisfies a level-3 large deviation principle with rate function* I^Φ.

REMARK 2.2. $\langle\nu,\Phi\rangle$ is called the *specific* (internal) *energy* and $I(\nu) + \langle\nu,\Phi\rangle$ the *specific free energy* of ν for Φ. $P(\Phi)$ is known as the *pressure* or *specific free Gibbs energy*. $I^\Phi(\nu)$ turns out to be the specific relative entropy of ν relative to an arbitrary $\mu\in\mathscr{G}(\Phi)$, namely,

$$I^\Phi(\nu) = \lim_{|\Lambda|\to\infty}|\Lambda|^{-1}I(\nu_\Lambda\colon\mu_\Lambda).$$

Thus $I^\Phi(\nu) = 0$ when $\nu\in\mathscr{G}_\Theta(\Phi)$. A celebrated variational principle of Lanford and Ruelle asserts that the converse is also true. Details can be found in Georgii (1988), Chapter 15.

PROOF OF THEOREM 2.1. Let us first look at the specific energy. Consider the function $f_\Phi = \sum_{A \ni 0} |A|^{-1} \Phi_A$. By (P1) and (P3), $f_\Phi \in \bar{\mathscr{L}}$, the $\|\cdot\|$-closure of $\mathscr{L}$. The functional $F\colon \nu \to \langle \nu, \Phi \rangle = \nu(f_\Phi)$ is therefore continuous, Also, a straightforward application of (P2) yields the estimate

$$
\begin{aligned}
&\sup_{\omega, \zeta \in \Omega} \left| |\Lambda|^{-1} H_\Lambda^{\Phi, \omega}(\zeta_\Lambda) - |\Lambda|^{-1} \sum_{i \in \Lambda} f_\Phi \circ \theta_i(\zeta) \right| \\
&\qquad \le \delta(\Lambda, \phi) := 2|\Lambda|^{-1} \sum_{i \in \Lambda} \sum_{A \ni 0} \sum_{A \setminus (\Lambda - i) \neq \varnothing} \|\Phi_A\|,
\end{aligned} \tag{2.3}
$$

and (P3) implies that $\delta(\Lambda, \Phi) \to 0$ as $|\Lambda| \to \infty$; see Georgii [(1988), page 320], for more details. This shows that for each $\nu \in \mathscr{P}_\Theta$,

$$\langle \nu, \Phi \rangle = \lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \nu_\Lambda\left(H_\Lambda^{\Phi, \omega}\right)$$

uniformly in $\omega \in \Omega$. Next, we observe that for every family $(\omega^\Lambda)_{\Lambda \in \mathscr{S}_c}$ in Ω,

$$\lim_{|\Lambda| \to \infty} \left\| |\Lambda|^{-1} H_\Lambda^{\Phi^\Lambda, \omega^\Lambda} - \langle \rho_\Lambda, \Phi \rangle \right\| = 0. \tag{2.4}$$

This follows from (2.3) as applied to Φ^Λ instead of Φ, Remark 0.1 and the obvious inequalities

$$\delta(\Lambda, \Phi^\Lambda) \le 2\|\Phi^\Lambda - \Phi\|_0 + \delta(\Lambda, \Phi), \qquad \|f_{\Phi^\Lambda} - f_\Phi\| \le \|\Phi^\Lambda - \Phi\|_0. \tag{2.5}$$

Equation (2.4) means that $\{(H_\Lambda^{\Phi^\Lambda, \omega^\Lambda}), \langle \,\cdot\,, \Phi \rangle\}$ is an asymptotic empirical functional. Theorem 1.2 thus implies that

$$\lim_{|\Lambda| \to \infty} |\Lambda|^{-1} \log Z_\Lambda^{\Phi^\Lambda, \omega^\Lambda} = -\inf[I + \langle \,\cdot\,, \Phi \rangle].$$

Finally, we have $\gamma_\Lambda^{\Phi^\Lambda, \omega^\Lambda} = \mu_\Lambda^F$ when $F_\Lambda = H_\Lambda^{\Phi^\Lambda, \omega^\Lambda}$. The theorem therefore follows from Corollary 1.5 by a suitable choice of (ω^Λ). □

2.2. *Thermodynamic formalism*: *an application of the contraction principle*. In this section we will show how to apply the theorem above to the thermodynamic formalism as set out by Lanford (1973). We begin by noting that the space $\mathscr{B}_\Theta$ of all potentials [in the sense of (P1) to (P3)] is a Banach space with norm $\|\cdot\|_0$. Let V be an arbitrary closed subspace of $\mathscr{B}_\Theta$. Its topological dual V^* is equipped with the weak* topology and the evaluation σ-algebra. [Recall that these are the smallest topology, resp., σ-algebra, on V^* making the evaluation maps $V^* \ni \tau \to \tau(\Psi)$ with $\Psi \in V$ continuous, resp., measurable.]

The well-known contraction principle of large deviation theory describes the behaviour of the large deviation principle under a continuous mapping; see, for example, Orey (1986) or Varadhan (1988). We shall apply the contraction principle to the affine continuous mapping $\varphi_V\colon \mathscr{P}_\Theta \to V^*$ defined by

$$\varphi_V(\nu)(\Psi) = \langle \nu, \Psi \rangle, \qquad \nu \in \mathscr{P}_\Theta, \Psi \in V. \tag{2.6}$$

Thus $\varphi_V(\nu)$ is the restriction of the specific energy functional $\langle \nu, \cdot \rangle$ on $\mathscr{B}_\Theta$ to V. Composing φ_V with the periodic empirical field ρ_Λ relative to a cube Λ we obtain the random functional $\tau_{\Lambda, V} = \varphi_V \circ \rho_\Lambda\colon E^\Lambda \to V^*$ in Λ. By definition, for each $\omega \in E^{\Lambda}$ and $\Psi \in V$ we have

$$\tau^{\omega}_{\Lambda,V}(\Psi) = \langle \rho^{\omega}_{\Lambda}, \Psi \rangle = |\Lambda|^{-1} H^{\Psi,\mathrm{per}}_{\Lambda}(\omega), \tag{2.7}$$

where

$$H^{\Psi,\mathrm{per}}_{\Lambda}(\omega) = \sum_{A} \sum_{i \in \Lambda} \Psi_{A+i}(\omega^{\mathrm{per}}), \qquad \omega \in E^{\Lambda} \tag{2.8}$$

is the Hamiltonian for Ψ in Λ with periodic boundary condition. In (2.8), the sum over A contains precisely one translate of every set in $\mathscr{S}$. Theorem 2.1 thus gives us the following uniform large deviation principle for the distribution of $\tau_{\Lambda,V}$ under Gibbs distributions.

Corollary 2.3. *Let V be a closed subspace of $\mathscr{B}_{\Theta}$, $\Phi \in \mathscr{B}_{\Theta}$, $(\Phi^{\Lambda})_{\Lambda \in \mathscr{S}_c}$ a family in $\mathscr{B}_{\Theta}$ such that $\|\Phi^{\Lambda} - \Phi\|_0 \to 0$ as $|\Lambda| \to \infty$, and K a measurable subset of V^*. Then*

$$\limsup_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \sup_{\omega \in \Omega} \gamma^{\Phi^{\Lambda},\omega}_{\Lambda}(\tau_{\Lambda,V} \in K) \le -\inf J^{\Phi}_{V}(\overline{K})$$

and

$$\liminf_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \inf_{\omega \in \Omega} \gamma^{\Phi^{\Lambda},\omega}_{\Lambda}(\tau_{\Lambda,V} \in K) \ge -\inf J^{\Phi}_{V}(K^{0}),$$

where the function $J^{\Phi}_{V}\colon V^* \to [0,\infty]$ is given by

$$\begin{aligned} J^{\Phi}_{V}(\tau) &= \inf I^{\Phi}(\varphi_V^{-1}\{\tau\}) \\ &= P(\Phi) + \inf_{\Psi \in V} [\tau(\Psi) + P(\Phi + \Psi)], \quad \tau \in V^*. \end{aligned} \tag{2.9}$$

J^{Φ}_{V} is convex and lower semicontinuous with compact level sets, and its effective domain $\{J^{\Phi}_{V} < \infty\}$ coincides with

$$D_V := \{\varphi_V(\nu) : \nu \in \mathscr{P}_{\Theta}, I(\nu) < \infty\}. \tag{2.10}$$

Proof. We only need to comment on (2.9). The basic observation is that for each $\tau \in V^*$, $\Psi \in V$ and $\nu \in \varphi_V^{-1}\{\tau\}$,

$$I^{\Phi}(\nu) = I^{\Phi+\Psi}(\nu) + P(\Phi) - P(\Phi + \Psi) - \tau(\Psi). \tag{2.11}$$

Since $I^{\Phi+\Psi} \ge 0$, we immediately obtain an inequality for the quantities in (2.9). The equality then follows from the Hahn–Banach theorem and Proposition (16.11) and Theorem (16.13) of Georgii (1988). Details can be found in Pirlot (1985). □

2.3. *The equivalence of ensembles.* Here we will apply the maximum entropy principle, Theorem 1.6, to the setting of the preceding subsection. For a given potential $\Phi \in \mathscr{B}_{\Theta}$ and cube Λ we consider the Gibbs distribution $\gamma^{\Phi,\mathrm{per}}_{\Lambda}$ with periodic boundary condition. $\gamma^{\Phi,\mathrm{per}}_{\Lambda}$ is defined as the probability measure on $(E,\mathscr{E})^{\Lambda}$ with λ^{Λ}-density

$$\exp[-H^{\Phi,\mathrm{per}}_{\Lambda}] / \lambda^{\Lambda}(\exp[-H^{\Phi,\mathrm{per}}_{\Lambda}]); \tag{2.12}$$

compare (2.8). The same symbol $\gamma_\Lambda^{\Phi,\,\mathrm{per}}$ will also be used to denote any element of $\mathscr{P}$ whose marginal distribution on E^Λ is given by (2.12). Conditioning $\gamma_\Lambda^{\Phi,\,\mathrm{per}}$ on the event that the energy functional $\tau_{\Lambda,V}$ takes a prescribed set of values, we obtain a periodic Gibbs distribution of microcanonical type. We will show that, under suitable hypotheses, all accumulation points of these microcanonical Gibbs distributions are Gibbs measures for a suitable potential. This is a general version of the well-known principle of equivalence of ensembles, as will be explained in more detail in Subsection 3.1. For $\Phi \in \mathscr{B}_\Theta$ and V as before we introduce the set

$$D_V^\Phi = \{\varphi_V(\nu): \nu \in \mathscr{G}_\Theta(\Phi + \Psi) \text{ for some } \Psi \in V\}. \tag{2.13}$$

In Lemma 4.9 we shall show that $\tau \in D_V^\Phi$ if and only if the convex function $V \ni \Psi \to \tau(\Psi) + P(\Phi + \Psi)$ on V attains its infimum or, equivalently, the function J_V^Φ admits a tangent functional at τ. In particular, if V is finite-dimensional, D_V^Φ contains the relative interior D_V^{ri} of D_V, that is, the interior of D_V relative to the smallest affine subspace of V^* containing D_V. In the general case we know, at least, that D_V^Φ is norm-dense in D_V relative to the usual norm on V^*; compare Lemma 4.10.

THEOREM 2.4. *Consider the setting of Corollary* 2.3. *Suppose K is convex and such that $K^0 \cap D_V \neq \varnothing$ and $K_{\min}^\Phi \subset D_V^\Phi$, where*

$$K_{\min}^\Phi = \{\tau \in \overline{K}: J_V^\Phi(\tau) = \inf J_V^\Phi(\overline{K})\}.$$

Then there exists some $\Psi \in V$ such that

$$\varnothing \neq \operatorname*{acc}_{\Lambda \uparrow S} \gamma_\Lambda^{\Phi^\Lambda,\,\mathrm{per}}(\,\cdot\,|\tau_{\Lambda,V} \in K) \subset \mathscr{G}_\Theta(\Phi + \Psi) \cap \varphi_V^{-1} K_{\min}^\Phi, \tag{2.14}$$

and each such Ψ is a point of minimum of the function $\tau + P(\Phi + \cdot\,)$ on V, for all $\tau \in K_{\min}^\Phi$. A similar conclusion holds for averaged [*in the sense of* (1.13)] *conditional Gibbs distributions with configurational boundary conditions.*

To deduce this result from Theorem 1.6 we need to show that the set M^F in (1.15), for the appropriate choice of F, is contained in $\mathscr{G}_\Theta(\Phi + \Psi)$ for some $\Psi \in V$. The assumption $K_{\min}^\Phi \subset D_V^\Phi$, though difficult to check in general, is clearly necessary for this to hold. In fact, it may be replaced by the weaker assumption that for some $\tau^* \in K_{\min}^\Phi \cap D_V^\Phi$ and all other $\tau \in K_{\min}^\Phi$, D_V^Φ contains a nontrivial convex mixture of τ^* and τ. This will be evident from the proof of Theorem 2.4 in subsection 4.3.

3. Special cases and examples.

3.1. *Large deviations and maximum entropy principle on level* 1. Let $N \geq 1$ be an integer and $\Psi^1, \dots, \Psi^N \in \mathscr{B}_\Theta$ be N potentials. We write $\boldsymbol{\Psi} = (\Psi^1, \dots, \Psi^N)$ for the associated $\mathbb{R}^N$-valued potential. We assume without loss of generality that $\Psi^1, \dots, \Psi^N$ are linearly independent and let V denote their linear span, that is,

$$V = \left\{ t \cdot \boldsymbol{\Psi} = \sum_{1}^{N} t_n \Psi^n : t \in \mathbb{R}^N \right\}.$$

For each cube Λ, the V^*-valued function $\tau_{\Lambda, V}$ in (2.7) can then be identified with the $\mathbb{R}^N$-valued rescaled periodic Hamiltonian

$$|\Lambda|^{-1} H_\Lambda^{\boldsymbol{\Psi}, \text{per}} = |\Lambda|^{-1} \left(H_\Lambda^{\Psi^1, \text{per}}, \dots, H_\Lambda^{\Psi^N, \text{per}} \right).$$

Corollary 2.3 thus gives the following result.

For each $\Phi \in \mathscr{B}_\Theta$ and all Borel sets $K \subset \mathbb{R}^N$,

$$(3.1) \qquad \left. \begin{matrix} \limsup\limits_{|\Lambda| \to \infty} & \sup\limits_{\omega \in \Omega} \\ \liminf & \inf \end{matrix} \right\} |\Lambda|^{-1} \log \gamma_\Lambda^{\Phi, \omega}\left(|\Lambda|^{-1} H_\Lambda^{\boldsymbol{\Psi}, \text{per}} \in K \right) \left\{ \begin{matrix} \le -\inf J_{\boldsymbol{\Psi}}^{\Phi}(\overline{K}) \\ \ge -\inf J_{\boldsymbol{\Psi}}^{\Phi}(K^0) \end{matrix} \right. ,$$

where $J_{\boldsymbol{\Psi}}^{\Phi}$ is given by

$$(3.2) \qquad J_{\boldsymbol{\Psi}}^{\Phi}(x) = P(\Phi) - \inf_{t \in \mathbb{R}^N} \left[t \cdot x + P(\Phi + t \cdot \boldsymbol{\Psi}) \right], \qquad x \in \mathbb{R}^N.$$

The effective domain $\{J_{\boldsymbol{\Psi}}^{\Phi} < \infty\}$ of $J_{\boldsymbol{\Psi}}^{\Phi}$ coincides with

$$(3.3) \qquad D_{\boldsymbol{\Psi}} = \left\{ \langle \nu, \boldsymbol{\Psi} \rangle : \nu \in \mathscr{P}_\Theta, I(\nu) < \infty \right\}.$$

This is Lanford's (1973) large deviation principle for rescaled Hamiltonians; see also Janžura (1985), Olla (1988) and Föllmer and Orey (1988). By Remark 1.4, (3.1) remains true for Hamiltonians with configurational (rather than periodic) boundary conditions, and all other asymptotically negligible perturbations of $H_\Lambda^{\Psi, \text{per}}$. In the special case when $N = 1$, $\Phi = 0$ and

$$\Psi_A = \begin{cases} f \circ X_i, & \text{if } A = \{i\} \text{ for some } i \in S, \\ 0, & \text{otherwise,} \end{cases}$$

for some measurable function $f \colon E \to \mathbb{R}$, (3.1) reduces to the well-known Cramér theorem because in this case $\gamma_\Lambda^{\Phi, \omega} = \lambda^\Lambda$, $H_\Lambda^{\Psi, \text{per}} = \sum_{i \in \Lambda} f \circ X_i$, and

$$J_\Psi^\Phi(x) = \sup_{t \in \mathbb{R}} \left[tx - \log \lambda(e^{tf}) \right], \qquad x \in \mathbb{R}.$$

Our next subject is the maximum entropy principle. In the present setting, Theorem 2.4 and the remarks before and after this theorem give us the following result.

Suppose K is a convex Borel set in $\mathbb{R}^N$ satisfying $K^0 \cap D_{\boldsymbol{\Psi}} \neq \varnothing$ and $K_{min}^{\Phi, \boldsymbol{\Psi}} \cap D_{\boldsymbol{\Psi}}^{ri} \neq \varnothing$, where

$$K_{\min}^{\Phi, \boldsymbol{\Psi}} = \left\{ x \in \overline{K} : J_{\boldsymbol{\Psi}}^{\Phi}(x) = \inf J_{\boldsymbol{\Psi}}^{\Phi}(\overline{K}) \right\}.$$

Then there exists some $t \in \mathbb{R}^N$ such that

$$(3.4) \qquad \begin{aligned} \varnothing &\neq \operatorname*{acc}_{\Lambda \uparrow S} \gamma_\Lambda^{\Phi, \text{per}}\left(\cdot \,\middle|\, |\Lambda|^{-1} H_\Lambda^{\boldsymbol{\Psi}, \text{per}} \in K \right) \\ &\subset \mathscr{G}_\Theta(\Phi + t \cdot \boldsymbol{\Psi}) \cap \left\{ \langle \cdot , \boldsymbol{\Psi} \rangle \in K_{\min}^{\Phi, \boldsymbol{\Psi}} \right\}, \end{aligned}$$

and each such t minimizes the function $t \to t \cdot x + P(\Phi + t \cdot \boldsymbol{\Psi})$ for all $x \in K_{\min}^{\Phi, \boldsymbol{\Psi}}$.

In the particular case $\Phi = 0$, $N = 1$, $\Psi^1 = \Psi \in \mathscr{B}_\Theta$, (3.4) just means that

$$(3.5)\quad \varnothing \neq \operatorname*{acc}_{\Lambda \uparrow S} \lambda^\Lambda\left(\cdot \mid |\Lambda|^{-1} H_\Lambda^{\Psi,\mathrm{per}} \in K\right) \subset \mathscr{G}_\Theta(\beta\Psi) \cap \left\{\langle \cdot , \Psi\rangle \in K_{\min}^{0,\Psi}\right\},$$

for some $\beta \in \mathbb{R}$ depending on Ψ and K. The conditional distributions in (3.5) are precisely the microcanonical Gibbs distributions (with periodic boundary condition) on the thick Ψ-energy shells described by the interval K. As the definition of $\mathscr{G}_\Theta(\beta\Psi)$ involves the grand canonical Gibbs distributions, assertion (3.5) can be paraphrased by saying that every microcanonical limiting equilibrium state for Ψ is a grand canonical equilibrium state for Ψ (or $-\Psi$) at a suitable inverse temperature $|\beta|$. Similarly, for a special choice of Ψ the conditional distributions in (3.4) take the form of the small canonical Gibbs distributions. Assertions (3.4) and (3.5) thus express the equivalence of the three Gibbs ensembles.

Three further remarks on (3.4) and (3.5) are in order. First, why do we insist on considering the relative interior of $D_{\boldsymbol{\Psi}}$ rather than its usual interior? The answer is simple: Although $\Psi^1, \dots, \Psi^N$ are supposed to be linearly independent, they may admit a nontrivial linear combination $\Psi = t \cdot \boldsymbol{\Psi}$ which is equivalent to 0, in that $\langle \cdot , \Psi\rangle$ is constant on $\{I < \infty\}$. In this case, $D_{\boldsymbol{\Psi}}$ is contained in a hyperplane and thus has empty interior. Lemma 4.11 will imply that this does not occur if and only if, for one and thus all $\Phi \in \mathscr{B}_\Theta$, P is strictly convex on $\Phi + V$, and in this case $D_{\boldsymbol{\Psi}}^0 \neq \varnothing$, and $J_{\boldsymbol{\Psi}}^\Phi$ is differentiable on $D_{\boldsymbol{\Psi}}^0$ with $|\operatorname{grad} J_{\boldsymbol{\Psi}}^\Phi(x)| \to \infty$ as $x \to \partial D_{\boldsymbol{\Psi}}$. The vector t in (3.4) then equals $-\operatorname{grad} J_{\boldsymbol{\Psi}}^\Phi(x)$ for each $x \in K_{\min}^{\Phi,\boldsymbol{\Psi}} \cap D_{\boldsymbol{\Psi}}^0$. Examples of spaces V on which P is strictly convex are given in Georgii [(1988), Corollary (16.15)].

The second remark concerns assertion (3.5). If Ψ is not equivalent to 0 and the interval K in (3.5) shrinks to some $x \in D_\Psi^0$, then it follows readily from the above that the associated parameter $\beta = \beta(K)$ converges to $\beta(x) = -(J_\Psi^0)'(x)$. It thus follows from (3.5) and Theorem 4.23(c) of Georgii (1988) that, in the double limit $\Lambda \uparrow S$ and $K \downarrow x$, every accumulation point of the microcanonical Gibbs distributions belongs to $\mathscr{G}_\Theta(\beta(x)\Psi)$. A weaker result of this type has been obtained recently by Deuschel, Stroock and Zessin (1991).

As for the third remark, we shall see in Example 3.2 that the inclusion in (3.5) can be strict. So it might seem that, in general, there exist more grand canonical equilibrium states than microcanonical equilibrium states. The following converse to (3.5) shows that this is not the case: Every Gibbs measure is a limit of microcanonical Gibbs distributions, at least if the energy shells are allowed to vary with Λ. Specifically, we have the following theorem which will be proved in subsection 4.4.

THEOREM 3.1. *For each $\Psi \in \mathscr{B}_\Theta$ and all $\mu \in \mathscr{G}_\Theta(\Psi)$ there exist sequences (Λ_k) in $\mathscr{S}_c$, (Ψ^k) in $\mathscr{B}_\Theta$ and (c_k) in $\mathbb{R}$ such that $\Lambda_k \uparrow S$, $\|\Psi^k - \Psi\|_0 \to 0$, $c_k \to \langle \mu, \Psi\rangle$ and*

$$\mu = \lim_{k\to\infty} \lambda^{\Lambda_k}\left(\cdot \mid |\Lambda_k|^{-1} H_{\Lambda_k}^{\Psi^k,\mathrm{per}} \le c_k\right).$$

As a matter of fact, we can extend (3.5), using the ideas of Remark 1.4, to show that every limit of microcanonical distributions as above necessarily belongs to $\mathscr{G}_\Theta(\Psi)$. Hence, $\mathscr{G}_\Theta(\Psi)$ coincides with the set of all these limits. We conclude this subsection with the previously mentioned example.

EXAMPLE 3.2 (The two-dimensional Ising model). Let $d = 2$, $E = \{-1, 1\}$, λ the equidistribution on E, and $\Psi \in \mathscr{B}_\Theta$ be defined by

$$\Psi_A(\omega) = \begin{cases} -\omega_i\omega_j, & \text{if } A = \{i, j\}, |i - j| = 1, \\ 0, & \text{otherwise}. \end{cases}$$

The function $\beta \to P(\beta\Psi)$ is even, as is easily seen by means of the reflection $\omega_i \to -\omega_i$ at every second site i of S. [In fact, the famous Onsager formula even provides an explicit expression for this function; see Georgii (1988), page 450, for references.] Equation (3.2) for $\Phi \equiv 0$ thus implies that the conjugate function $J_\Psi = J_\Psi^0$ is also even. J_Ψ thus attains its minimum 0 at $x = 0$ and is decreasing on $]-\infty, 0]$. It is also easy to see that $D_\Psi = [-2, 2]$. J_Ψ is differentiable on $]-2, 2[$ with $|J_\Psi'(x)| \to \infty$ for $|x| \uparrow 2$; compare the first remark on (3.4) above. Further, for our choice of Ψ all accumulation points in (3.5) necessarily share the invariance of λ and Ψ under the reflection of E, and it is known that for each $\beta \geq 0$ there exists a unique reflection-invariant $\mu^\beta \in \mathscr{G}_\Theta(\beta\Psi)$; compare the references on pages 453 and 472 of Georgii (1988). Equation (3.5) thus implies the following result. For any $c \in]-2, 2[$,

$$\lim_{\Lambda \uparrow S} \lambda^\Lambda\left(\cdot \mid |\Lambda|^{-1} H_\Lambda^{\Psi, \mathrm{per}} \leq c\right) = \mu^{\beta(c)},$$

where $\beta(c) = -J_\Psi'(c) > 0$ for $c < 0$ and $\beta(c) = 0$ for $c \geq 0$. On the other hand, if $\beta = \beta(c)$ is sufficiently large, that is, if c is sufficiently close to -2, $\mathscr{G}_\Theta(\beta\Psi)$ is not a singleton, but $\langle \cdot, \Psi \rangle$ is constant on $\mathscr{G}_\Theta(\beta\Psi)$; compare Theorem (6.9) of Georgii (1988). This shows that the inclusion in (3.5) can be strict. Some further, related results for this model can be found in Deuschel, Stroock and Zessin (1991).

3.2. *Large deviations and maximum entropy principle on level* 2. Here we apply the results of subsections 2.2 and 2.3 to the space

$$V = \{\Psi \in \mathscr{B}_\Theta: \Psi_A = 0 \text{ unless } |A| = 1\} \tag{3.6}$$

of all self-potentials in $\mathscr{B}_\Theta$. (More generally, we could consider all Ψ with $\Psi_A = 0$ except when A is a translate of a given base $\Delta \in \mathscr{S}$.) We note first that the relation

$$\Psi_{\{i\}}(\omega) = f(\omega_i), \qquad \omega \in \Omega, i \in S, \tag{3.7}$$

establishes a one-to-one correspondence between the potentials $\Psi \in V$ and the bounded measurable functions f on $(E, \mathscr{E})$. Equation (3.7) implies that $\langle \nu, \Psi \rangle = \nu_{\{0\}}(f)$ for all $\nu \in \mathscr{P}_\Theta$. Consequently, the mapping φ_V in (2.6) is nothing other than the marginal projection $\nu \to \nu_{\{0\}}$ from $\mathscr{P}_\Theta$ into the subset $\mathscr{P}(E, \mathscr{E})$ of V^*. The specific energy function $\tau_{\Lambda, V}$ relative to a cube Λ thus coincides with the *empirical distribution*

$$\pi_\Lambda^\omega = |\Lambda|^{-1} \sum_{i \in \Lambda} \delta_{\omega_i}, \qquad \omega \in E^\Lambda. \tag{3.8}$$

As a further consequence of (3.7), the formula (2.9) for the rate function J_V^Φ can be rewritten as

$$\begin{aligned} J^\Phi(\alpha) &= \inf_{\nu \in \mathscr{P}_\Theta:\, \nu_{\{0\}} = \alpha} I^\Phi(\nu) \\ &= P(\Phi) - \inf_f \left[\alpha(f) + P(\Phi + \Psi^f) \right], \qquad \alpha \in \mathscr{P}(E, \mathscr{E}). \end{aligned} \tag{3.9}$$

In (3.9), the infimum over f extends over all bounded measurable functions on $(E, \mathscr{E})$, and Ψ^f is the potential in V related to f via (3.7). We finally note that the weak* topology on V^* induces the τ-topology on $\mathscr{P}(E, \mathscr{E})$. It is thus obvious how Corollary 2.3 and Theorem 2.4 can be restated for the present choice of V. We refrain from doing so; the case of finite E is treated in detail in Föllmer and Orey (1988). In the case $\Phi = 0$ we obtain the classical Sanov's theorem in the version of Groeneboom, Oosterhoff and Ruymgaart (1979) and a conditional limit theorem of Csiszár's (1984) type. Equation (3.9) then reduces to the well-known formula

$$I(\alpha; \lambda) = \sup_f \left[\alpha(f) - \log \lambda(e^f) \right],$$

a direct proof of which can be found in Varadhan (1988), for example.

Another case of classical interest is when $d = 1$ and Φ is a bounded nearest-neighbour potential. Corollary 2.3, for the space V in (3.6), then gives the level-2 large deviation principle for uniformly recurrent Markov chains in the version of Bolthausen (1987). Moreover, the rightmost expression in (3.9) then can be shown to coincide with Donsker and Varadhan's (1975, 1976) rate function $\sup \alpha(\log u/Qu)$; here the sup extends over all positive measurable functions u on E which are bounded away from 0 and ∞, and Q is the transition kernel of the Markov chain which is the unique element of $\mathscr{G}(\Phi)$.

4. Proofs.

4.1. *The upper bound.* In this section we will prove the first half of Theorem 1.2, namely inequality (1.8). We thus need to find an upper estimate of the quantity

$$|\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}). \tag{4.1}$$

Here $\{(F_\Lambda), F\}$ is a given asymptotic empirical functional, and Λ is an arbitrary cube. We can assume without loss of generality that $F \geq 0$. Clearly, we can also assume that $\lambda^\Lambda(e^{-F_\Lambda}) > 0$. We then can define the associated Gibbs distribution μ_Λ^F; compare (1.12). The relative entropy of μ_Λ^F relative to λ^Λ is related to the expression (4.1) via the key identity

$$-|\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) = |\Lambda|^{-1} I(\mu_\Lambda^F; \lambda^\Lambda) + |\Lambda|^{-1} \mu_\Lambda^F(F_\Lambda), \tag{4.2}$$

as is easily checked using (1.1) and (1.12). This is an observation of Csiszár (1984). We therefore proceed by deriving lower bounds for the terms on the right-hand side of (4.2). To deal with the first term we introduce the measure

$$\tilde{\mu}_\Lambda^F = |\Lambda|^{-1} \sum_{i \in \Lambda} \left(\mu_\Lambda^F\right)^{pS} \circ \theta_i^{-1}. \tag{4.3}$$

Here p is the side length of Λ [compare (0.3)] and $(\mu_\Lambda^F)^{pS}$ stands for the probability measure on $(\Omega, \mathscr{F})$ relative to which the projections $(X_{\Lambda+pi})_{i \in S}$ are independent with identical distribution μ_Λ^F. [It should be noted that the construction (4.3) can be traced back to Parthasarathy (1961).]

LEMMA 4.1. *For each cube* Λ, $\tilde{\mu}_\Lambda^F \in \mathscr{P}_\Theta$ *and*

$$I\left(\tilde{\mu}_\Lambda^F\right) = |\Lambda|^{-1} I\left(\mu_\Lambda^F; \lambda^\Lambda\right).$$

PROOF. The first assertion is an obvious consequence of the Λ-periodicity of $(\mu_\Lambda^F)^{pS}$. A proof of the second assertion can be found in Georgii (1988), Proposition (16.34), for example. □

To estimate the second term on the right-hand side of (4.2) we introduce the lower convex envelope $\underline{F}$ of F. By definition,

$$\underline{F}(\mu) = \sup\{G(\mu) \colon G \in \mathscr{A}, G < F\}, \qquad \mu \in \mathscr{P}_\Theta, \tag{4.4}$$

where $\mathscr{A}$ consists of the affine continuous evaluation functions $\mathscr{P}_\Theta \ni \nu \to \nu(g)$ with $g \in \mathscr{L}$. As $F \geq 0$, $\underline{F}$ is well defined and nonnegative. Clearly, $\underline{F}$ is lower semicontinuous. (A different expression for $\underline{F}$ will be derived in Lemma 4.7.)

LEMMA 4.2. *For each cube* Λ,

$$|\Lambda|^{-1} \mu_\Lambda^F(F_\Lambda) \geq \underline{F}\left(\mu_\Lambda^F \rho_\Lambda\right) + \inf\left[|\Lambda|^{-1} F_\Lambda - F(\rho_\Lambda)\right],$$

where $\mu_\Lambda^F \rho_\Lambda \in \mathscr{P}_\Theta$ *is defined by* $\mu_\Lambda^F \rho_\Lambda = \int \mu_\Lambda^F(d\omega) \rho_\Lambda^\omega$.

PROOF. Let $g \in \mathscr{L}$ be such that $\nu(g) < F(\nu)$ for all $\nu \in \mathscr{P}_\Theta$. Then

$$|\Lambda|^{-1} F_\Lambda \geq \rho_\Lambda(g) + \inf\left[|\Lambda|^{-1} F_\Lambda - F(\rho_\Lambda)\right].$$

Integrating this inequality with respect to μ_Λ^F and taking the supremum over all such g, we obtain the result. □

The next step is to show that the measures $\tilde{\mu}_\Lambda^F$ and $\mu_\Lambda^F \rho_\Lambda$ in Lemmas 4.1 and 4.2 have the same set of accumulation points as $|\Lambda| \to \infty$.

LEMMA 4.3. *For all* $f \in \mathscr{L}$,

$$\lim_{|\Lambda| \to \infty} \left[\tilde{\mu}_\Lambda^F(f) - \mu_\Lambda^F \rho_\Lambda(f)\right] = 0.$$

PROOF. We pick any $\Delta \in \mathscr{S}$ such that f is $\mathscr{F}_\Delta$-measurable. Since $f \circ \theta_i$ is $\mathscr{F}_\Lambda$-measurable for any $i \in \Lambda$ with $\Delta + i \subset \Lambda$, the difference under consideration equals

$$|\Lambda|^{-1} \sum_{i \in \Lambda:\, \Delta + i \not\subset \Lambda} \left[\left(\mu_\Lambda^F\right)^{pS} (f \circ \theta_i) - \int \mu_\Lambda^F(d\omega)\, f \circ \theta_i(\omega^{\mathrm{per}}) \right].$$

This is bounded in modulus by

$$2\|f\|\,|\Lambda|^{-1} |\{i \in \Lambda\colon \Delta + i \not\subset \Lambda\}|,$$

which tends to zero as $|\Lambda| \to \infty$. □

We now take advantage of the fact that I has compact level sets.

LEMMA 4.4. $\liminf_{|\Lambda| \to \infty} [I(\tilde{\mu}_\Lambda^F) + \underline{F}(\mu_\Lambda^F \rho_\Lambda)] \geq \inf[I + \underline{F}]$.

PROOF. Suppose the contrary. Then there exists a number $c < \inf[I + \underline{F}]$ and a sequence (Λ_k) of cubes such that $|\Lambda_k| \to \infty$ and

$$I\left(\tilde{\mu}_{\Lambda_k}^F\right) + \underline{F}\left(\mu_{\Lambda_k}^F \rho_{\Lambda_k}\right) \leq c$$

for all k. Since $\underline{F} \geq 0$, the sequence $(\tilde{\mu}_{\Lambda_k}^F)$ belongs to the (sequentially) compact set $\{I \leq c\}$ and thus admits a convergent subnet (even: subsequence) $(\tilde{\mu}_{\Lambda_l}^F)$ with limit ν, say. By Lemma 4.3, ν is also the limit of $(\mu_{\Lambda_l}^F \rho_{\Lambda_l})$. Since I and $\underline{F}$ are lower semicontinuous, we conclude that

$$I(\nu) + \underline{F}(\nu) \leq \liminf_l \left[I\left(\tilde{\mu}_{\Lambda_l}\right) + \underline{F}(\mu_{\Lambda_l} \rho_{\Lambda_l}) \right] \leq c,$$

in contradiction to the choice of c. □

Combining (4.2) with Lemmas 4.1, 4.2 and 4.4 and hypothesis (1.7) we obtain

$$\limsup_{|\Lambda| \to \infty} |\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) \leq -\inf[I + \underline{F}]. \tag{4.5}$$

To complete the proof of the upper bound (1.8) we thus need to replace $\underline{F}$ by F_{lsc} in the last infimum. This is possible because I is affine with compact level sets.

LEMMA 4.5. $\inf[I + \underline{F}] = \inf[I + F_{\mathrm{lsc}}]$.

PROOF. We proceed in three steps, using techniques of convex analysis.

(i) Let $\mathscr{L}^*$ be the topological dual of $\mathscr{L}$ and

$$\begin{aligned} \mathscr{K} = \{\nu \in \mathscr{L}^*\colon \nu(f) \geq 0 \text{ when } 0 \leq f \in \mathscr{L}, \\ \nu(1) = 1,\ \nu(f \circ \theta_i) = \nu(f) \text{ for all } f \in \mathscr{L},\ i \in S\} \end{aligned} \tag{4.6}$$

be the convex set of all shift-invariant, normalized, positive linear functionals on $\mathscr{L}$. $\mathscr{K}$ is a closed subset of the unit ball of $\mathscr{L}^*$. It thus follows from the Banach–Alaoglu theorem that $\mathscr{K}$ is compact in the weak* topology on $\mathscr{L}^*$.

The set $\mathscr{P}_\Theta$ can be regarded as the set of all $\nu \in \mathscr{K}$ satisfying

$$\nu(f_n) \downarrow 0 \text{ for every sequence } (f_n) \text{ in } \mathscr{L} \text{ with } f_n \downarrow 0. \tag{4.7}$$

Indeed, this condition implies that $A \to \nu(1_A)$ is σ-additive on the algebra of all cylinder events. The Caratheodory extension theorem thus shows that ν can be extended to a unique element of $\mathscr{P}_\Theta$. Therefore, we will think of $\mathscr{P}_\Theta$ as a subset of $\mathscr{K}$. The topology on $\mathscr{P}_\Theta$ induced by the weak* topology on $\mathscr{K}$ is precisely $\tau_{\mathscr{L}}$, the topology of local convergence.

As a matter of fact, $\mathscr{P}_\Theta$ is a face of $\mathscr{K}$. For let $\nu_1, \nu_2 \in \mathscr{K}$, $0 < s < 1$ and $\nu = s\nu_1 + (1-s)\nu \in \mathscr{P}_\Theta$. Property (4.7) for ν then implies the same property for ν_1 and ν_2. Hence $\nu_1, \nu_2 \in \mathscr{P}_\Theta$.

(ii) We extend the specific entropy I to a functional on $\mathscr{K}$ by setting $I(\nu) = \infty$ if $\nu \in \mathscr{K} \setminus \mathscr{P}_\Theta$. Since $\mathscr{P}_\Theta$ is a face of $\mathscr{K}$, this extension is still affine. For each $c \geq 0$, the level set $\{I \leq c\}$ is a compact subset of $\mathscr{P}_\Theta$ and thus a closed subset of $\mathscr{K}$. Therefore, the extended I is still lower semicontinuous. As a consequence of these properties, the separating hyperplane theorem implies that the family $\mathscr{A}_I$ of all functions G: $\nu \to \nu(g)$ on $\mathscr{K}$ with $g \in \mathscr{L}$ and $G < I$ is directed upward and satisfies

$$I = \sup_{G \in \mathscr{A}_I} G; \tag{4.8}$$

see Choquet [(1969), Theorem 21.21], or Phelps [(1966), page 68], and note that each continuous linear functional on $\mathscr{L}^*$ has the form $\nu \to \nu(g)$ with $g \in \mathscr{L}$.

(iii) Suppose now $m := \inf[I + \underline{F}]$ is finite and let $c > m$. For each $G \in \mathscr{A}_I$ we set

$$C_G = \{\nu \in \mathscr{P}_\Theta : G(\nu) + F_{\mathrm{lsc}}(\nu) \leq c\} \tag{4.9}$$

and let $\overline{C}_G$ denote its closure in $\mathscr{K}$. We have $C_G \neq \varnothing$ because otherwise $c - G$ (restricted to $\mathscr{P}_\Theta$) is a minorant of F and therefore also of $\underline{F}$, whence $I + \underline{F} \geq c$ in contradiction to the choice of c. Since $\mathscr{A}_I$ is directed upward and the sets $\overline{C}_G$ are compact and decreasing in G, it follows that

$$\bigcap_{G \in \mathscr{A}_I} \overline{C}_G \neq \varnothing.$$

But

$$\bigcap_{G \in \mathscr{A}_I} \overline{C}_G \subset \bigcap_{G \in \mathscr{A}_I} \{G \leq c\} = \{I \leq c\} \subset \mathscr{P}_\Theta$$

because $F_{\mathrm{lsc}} \geq 0$, and $C_G = \overline{C}_G \cap \mathscr{P}_\Theta$ because F_{lsc} is lower semicontinuous. Hence

$$\{I + F_{\mathrm{lsc}} \leq c\} = \bigcap_{G \in \mathscr{A}_I} C_G \neq \varnothing.$$

As $c > m$ was arbitrary, we conclude that $\inf[I + F_{\mathrm{lsc}}] \le m$. The reverse inequality is trivial because $\underline{F} \le F_{\mathrm{lsc}}$. $\square$

An alternative proof of this lemma will be given at the end of subsection 4.2. We conclude this subsection with the proof of Remark 1.4.

PROOF OF REMARK 1.4. We only derive the upper bound (1.8) under hypothesis (1.10). The proof of (1.9) under (1.11) is similar but simpler. For each $U \in \mathscr{U}$ we define $F_U\colon \mathscr{P}_\Theta \to]-\infty, \infty]$ by $F_U(\nu) = \inf F(U(\nu))$, $\nu \in \mathscr{P}_\Theta$. As we have seen above, (1.10) is all that is needed to derive (1.8) with F_U in place of F. Hence, the left-hand side of (1.8) is not larger than the negative of

$$\sup_{U \in \mathscr{U}} \inf\left[I + (F_U)_{\mathrm{lsc}}\right].$$

But this expression equals $\inf[I + F_{\mathrm{lsc}}]$. Indeed, otherwise we could find a number $c < \inf[I + F_{\mathrm{lsc}}]$ and, for each $U \in \mathscr{U}$, some $\nu_U \in \mathscr{P}_\Theta$ such that $I(\nu_U) + (F_U)_{\mathrm{lsc}}(\nu_U) < c$. This implies that for each U there exists some $\nu'_U \in U^2(\nu_U)$ with $I(\nu_U) + F(\nu'_U) < c$. Here, $U^2 \in \mathscr{U}$ is defined by doubling the δ defining U. Since F is bounded from below, the net (ν_U) belongs to a level set of I and therefore admits a convergent subnet with limit ν, say. The corresponding subnet of (ν'_U) also converges to ν. By lower semicontinuity, it follows that $I(\nu) + F_{\mathrm{lsc}}(\nu) \le c$, in contradiction to the choice of c. $\square$

4.2. *The maximum entropy principle.* In this section we will prove Theorem 1.6. We assume again without loss that $F \ge 0$. As a consequence of (1.14) and (1.9), μ_Λ^F is well defined for sufficiently large Λ. The key to the maximum entropy principle is a combination of the main ingredients of the proof of the upper bound with the lower bound, namely Lemmas 4.1 and 4.2, equation (4.2), the lower bound (1.9), assumption (1.14) and Lemma 4.5:

$$\begin{aligned}\limsup_{|\Lambda|\to\infty}\left[I(\tilde{\mu}_\Lambda^F) + \underline{F}(\mu_\Lambda^F \rho_\Lambda)\right] &\le -\liminf_{|\Lambda|\to\infty} |\Lambda|^{-1} \log \lambda^\Lambda(e^{-F_\Lambda}) \\ &\le \inf[I + F^{\mathrm{usc}}] = \inf[I + \underline{F}] < \infty.\end{aligned} \tag{4.10}$$

As $\underline{F} \ge 0$, this inequality implies that $(\tilde{\mu}_\Lambda^F)$ eventually belongs to a level set of I and thus admits at least one accumulation point μ. By the lower semicontinuity of I and $\underline{F}$ and Lemma 4.3, (4.10) implies further that

$$I(\mu) + \underline{F}(\mu) \le \inf[I + \underline{F}]$$

for each such μ. Hence

$$\varnothing \ne \operatorname*{acc}_{\Lambda \uparrow S} \tilde{\mu}_\Lambda^F \subset \{I + \underline{F} = \min\}. \tag{4.11}$$

The proof of Theorem 1.6 will thus be complete once we have shown that

(i) $\mathrm{acc}_{\Lambda \uparrow S}\, \tilde{\mu}_\Lambda^F = \mathrm{acc}_{\Lambda \uparrow S}\, \bar{\mu}_\Lambda^F$ and $\mathrm{acc}_{\Lambda \uparrow S}\, \tilde{\mu}_\Lambda^F = \mathrm{acc}_{\Lambda \uparrow S}\, \mu_\Lambda^F$ when each F_Λ is a function of ρ_Λ;

(ii) $\{I + \underline{F} = \min\}$ is equal to the set of all mixtures of measures in $\{I + F_{\mathrm{lsc}} = \min\}$.

Problem (i) is settled by the following analogue of Lemma 4.3.

LEMMA 4.6. *For each* $f \in \mathscr{L}$, $\lim_{|\Lambda| \to \infty} [\tilde{\mu}_\Lambda^F(f) - \bar{\mu}_\Lambda^F(f)] = 0$. *If each* F_Λ *is a functional of* ρ_Λ, *then also* $\lim_{\Lambda \uparrow S} [\tilde{\mu}_\Lambda^F(f) - \mu_\Lambda^F(f)] = 0$.

PROOF. In view of (1.13),

$$\bar{\mu}_\Lambda^F(f) = \mu_\Lambda^F\left(|\Lambda|^{-1} \sum_{i \in \Lambda} f \circ \theta_i\right).$$

The first assertion thus follows directly from Remark 0.1 and Lemma 4.3. Suppose next that each F_Λ is a function of ρ_Λ. In fact, this amounts to the requirement that each F_Λ is invariant under periodic shifts $\theta_i^{\mathrm{per}}\colon E^\Lambda \to E^\Lambda$ defined by

$$\theta_i^{\mathrm{per}}\omega = (\theta_i \omega^{\mathrm{per}})_\Lambda, \qquad \omega \in E^\Lambda, i \in \Lambda.$$

Indeed, the function $\omega \to \rho_\Lambda^\omega$ clearly has this invariance property. Conversely, if $g\colon E^\Lambda \to \mathbb{R}$ is invariant under $(\theta_i^{\mathrm{per}})_{i \in \Lambda}$, then

$$g(\omega) = |\Lambda|^{-1} \sum_{i \in \Lambda} g \circ \theta_i^{\mathrm{per}}(\omega) = \rho_\Lambda^\omega(g \circ X_\Lambda), \qquad \omega \in \Omega.$$

By (1.12), the measure μ_Λ^F inherits the invariance under $(\theta_i^{\mathrm{per}})_{i \in \Lambda}$. Thus, for each bounded measurable function g on E^Λ we have

$$\mu_\Lambda^F(g) = \mu_\Lambda^F\left(|\Lambda|^{-1} \sum_{i \in \Lambda} g \circ \theta_i^{\mathrm{per}}\right) = \mu_\Lambda^F \rho_\Lambda(g \circ X_\Lambda).$$

If we think again of μ_Λ^F as an element of $\mathscr{P}$ with Λ-marginal (1.12), this may be restated as follows. For each $f \in \mathscr{L}$, $\mu_\Lambda^F(f) = \mu_\Lambda^F \rho_\Lambda(f)$ provided Λ is so large that f is $\mathscr{F}_\Lambda$-measurable. The second assertion thus also follows from Lemma 4.3. □

Turning to problem (ii), we must show that each $\mu \in \{I + \underline{F} = \min\}$ admits a representation $\mu = \int w(d\nu)\nu$ in terms of a Borel probability measure w on $\{I + F_{\mathrm{lsc}} = \min\}$. More explicitly, this representation means that

$$\mu(A) = \int w(d\nu)\nu(A) \quad \text{for all } A \in \mathscr{F}.$$

This makes sense because the set of all A for which $A \to \nu(A)$ is Borel measurable is a monotone class containing the algebra of local events, and thus is equal to $\mathscr{F}$. A monotone class argument also shows that the representation above is equivalent to the requirement that

$$\mu(f) = \int w(d\nu)\nu(f) \quad \text{for all } f \in \mathscr{L}.$$

We need again a piece of convex analysis.

LEMMA 4.7. *Let $\mu \in \mathscr{P}_\Theta$ be such that $I(\mu) < \infty$, and $\mathscr{W}(\mu)$ the set of all regular Borel probability measures w on $\{I < \infty\}$ representing μ in the sense above. Then*:

(a) $I(\mu) = w(I)$ *for all* $w \in \mathscr{W}(\mu)$;

(b) $\underline{F}(\mu) = \inf\{w(F_{\mathrm{lsc}}): w \in \mathscr{W}(\mu)\}$, *and the infimum is attained when* $\underline{F}(\mu) < \infty$.

PROOF. (i) Consider the compact set $\mathscr{K}$ in (4.6) and the set $\mathscr{W}$ of all regular Borel probability measures on $\mathscr{K}$. $\mathscr{W}$ is compact in the weak topology. As shown in Step (b) of the proof of Lemma 4.5, I can be extended to an affine semicontinuous functional on $\mathscr{K}$. Thus, if $w \in \mathscr{W}$ represents μ, then Lemma 9.7 of Phelps (1966) implies that $I(\mu) = w(I)$. [Since each continuous linear functional on $\mathscr{L}^*$ has the form $\nu \to \nu(g)$ for some $g \in \mathscr{L}$ and $\nu(1) = 1$ for all $\nu \in \mathscr{K}$, Phelps' set A can be replaced by the set of all functions $\mathscr{K} \ni \nu \to \nu(g)$ with $g \in \mathscr{L}$.] This gives (a) because each Borel probability measure on the K_σ set $\{I < \infty\}$ can be considered as an element of $\mathscr{W}$.

(ii) To prove (b) we extend F to a function on $\mathscr{K}$ by setting $F = \infty$ on $\mathscr{K} \setminus \mathscr{P}_\Theta$, and we let F_* denote the lower semicontinuous regularization of F on $\mathscr{K}$. A glance at (1.6) shows that $F_{\mathrm{lsc}} = F_* | \mathscr{P}_\Theta$. As $\mathscr{K}$ is compact and thus completely regular, F_* is the supremum of all its continuous minorants. Hence

$$w(F_*) = \sup\{w(G): G \text{ continuous}, G < F_*\}$$

for all $w \in \mathscr{W}$; compare Phelps (1966), page 63. It follows that the mapping $w \to w(F_*)$ on $\mathscr{W}$ is lower semicontinuous. This in turn allows us to extend the proof of Proposition 3.1 of Phelps (1966) to show that

$$\underline{F}(\mu) = \inf\left\{w(F_*): w \in \mathscr{W}, \qquad \mu = \int w(d\nu)\nu\right\}$$

for all $\mu \in \mathscr{K}$. Now, if $I(\mu) < \infty$ then assertion (a) implies that $w(I) < \infty$ for each w contributing to the infimum. Hence each such w is supported on $\{I < \infty\}$. In particular, $w(F_*) = w(F_{\mathrm{lsc}})$. This gives the first part of (b). To obtain the second part it is sufficient to note that the set of all $w \in \mathscr{W}$ representing μ is compact and $w \to w(F_*)$ is lower semicontinuous. □

As an immediate consequence we obtain the following lemma which completes the proof of Theorem 1.6.

LEMMA 4.8. *Suppose that* $m := \inf[I + F_{\mathrm{lsc}}] < \infty$ *and let* $\mu \in \mathscr{P}_\Theta$. *Then* $I(\mu) + \underline{F}(\mu) = m$ *if and only if* μ *is represented by a regular Borel probability measure on the compact set* $\{I + F_{\mathrm{lsc}} = m\}$.

PROOF. By Lemma 4.5, $I(\mu) + \underline{F}(\mu) = m$ if and only if $I(\mu) + \underline{F}(\mu) \le m$. In view of Lemma 4.7, the latter holds if and only if there exists some $w \in \mathscr{W}(\mu)$ with $w(I + F_{\mathrm{lsc}}) \le m$. But this inequality means that w is supported on $\{I + F_{\mathrm{lsc}} = m\}$. □

To conclude this subsection we note that Lemma 4.7 provides an alternative proof of Lemma 4.5. Indeed, let $\mu \in \mathscr{P}_\Theta$ be such that $I(\mu) + \underline{F}(\mu) < \infty$. Then μ is represented by some w with $\underline{F}(\mu) = w(F_{\mathrm{lsc}})$. Hence

$$I(\mu) + \underline{F}(\mu) = w(I + F_{\mathrm{lsc}}) \ge \inf[I + F_{\mathrm{lsc}}].$$

Since $\underline{F} \le F_{\mathrm{lsc}}$, Lemma 4.5 follows.

4.3. *Equilibria under energy constraints.* Here we shall prove Theorem 2.4. So we let V be a closed subspace of $\mathscr{B}_\Theta$ and $\Phi \in \mathscr{B}_\Theta$ a potential. We shall drop the lower indices of φ_V and J_V^Φ. We consider the sets

$$\mathscr{C}^\Phi(\tau) = \{\nu \in \varphi^{-1}\{\tau\} : I^\Phi(\nu) = J^\Phi(\tau)\}, \qquad \tau \in V^*.$$

Since I^Φ has compact level sets, $\mathscr{C}^\Phi(\tau) \ne \varnothing$ when $\tau \in D$. The lemma below shows that, for a large set of τ's, $\mathscr{C}^\Phi(\tau)$ consists of Gibbs measures relative to $\Phi + \Psi$ for a suitable $\Psi \in V$, and provides various characterizations of this set of τ's. Consider the function $P^\Phi \colon V \to \mathbb{R}$ defined by $P^\Phi(\Psi) = P(\Phi + \Psi) - P(\Phi)$, $\Psi \in V$. By (2.9), P^Φ is just the convex conjugate of J^Φ relative to the bilinear form $(\tau, \Psi) \to -\tau(\Psi)$. A functional $\tau \in V^*$ is called a *tangent* to P^Φ at $\Psi \in V$ if $P^\Phi(\Psi) + \tau(X) \le P^\Phi(\Psi + X)$ for all $X \in V$. The set $\partial P^\Phi(\Psi)$ of all these tangents is called the *subdifferential* of P^Φ at Ψ. Similarly, the *subdifferential* $\partial J^\Phi(\tau)$ of J^Φ at $\tau \in V^*$ is defined as the set of all $\Psi \in V$ such that $J^\Phi(\tau) + \sigma(\Psi) \le J^\Phi(\tau + \sigma)$ for all $\sigma \in V^*$.

LEMMA 4.9. *For each* $\Phi \in \mathscr{B}_\Theta$, $\Psi \in V$ *and* $\tau \in V^*$ *the following statements are equivalent*:

(i) $\tau = \varphi(\nu)$ *for some* $\nu \in \mathscr{G}_\Theta(\Phi + \Psi)$.
(ii) $\tau + P^\Phi$ *reaches its infimum at* Ψ.
(iii) $-\tau \in \partial P^\Phi(\Psi)$.
(iv) $-\Psi \in \partial J^\Phi(\tau)$.
(v) $\tau \in D$, $\mathscr{C}^\Phi(\tau) \subset \mathscr{G}_\Theta(\Phi + \Psi)$.

Moreover, if $\tau \in D$ *and* $\Psi, \Psi' \in V$ *are such that* (i) *through* (v) *hold for both* (τ, Ψ) *and* (τ, Ψ'), *then* $\mathscr{G}(\Phi + \Psi) = \mathscr{G}(\Phi + \Psi')$.

PROOF. (i) $\Rightarrow$ (ii) Since $I^{\Phi+\Psi}(\nu) = 0$ when (i) holds, we conclude from (2.11) and (2.9) that $\tau(\Psi) + P^{\Phi}(\Psi) = -I^{\Phi}(\nu) \le -J^{\Phi}(\tau) = \inf[\tau + P^{\Phi}]$.

(ii) $\Leftrightarrow$ (iii) This follows right from the definition of $\partial P^{\Phi}(\Psi)$.

(ii) $\Rightarrow$ (iv) Equation (2.9) and (ii) imply that $J^{\Phi}(\tau) = -\tau(\Psi) - P^{\Phi}(\Psi) < \infty$ and $J^{\Phi}(\tau + \sigma) \ge -(\tau + \sigma)(\Psi) - P^{\Phi}(\Psi)$ for all σ. Subtracting the equality from the inequality we obtain the result.

(iv) $\Rightarrow$ (v) If $\partial J^{\Phi}(\tau) \ne \varnothing$ then clearly $J^{\Phi}(\tau) < \infty$ and thus $\tau \in D$. Suppose now that $-\Psi \in \partial J^{\Phi}(\tau)$ and let $\mu \in \mathscr{C}^{\Phi}(\tau)$ be given. We also pick an arbitrary $\nu \in \mathscr{G}_{\Theta}(\Phi + \Psi)$ and set $\sigma = \varphi(\nu) - \tau$. Then

$$I^{\Phi}(\mu) + \langle \mu - \nu, \Psi \rangle = J^{\Phi}(\tau) + \sigma(-\Psi) \le J^{\Phi}(\tau + \sigma) \le I^{\Phi}(\nu)$$

and thus $I^{\Phi+\Psi}(\mu) \le I^{\Phi+\Psi}(\nu) = 0$. By Remark 2.2, it follows that $\mu \in \mathscr{G}_{\Theta}(\Phi + \Psi)$.

(v) $\Rightarrow$ (i) This is trivial.

Finally, suppose (i) to (v) hold for the pairs (τ, Ψ) and (τ, Ψ'). Since $\mathscr{C}^{\Phi}(\tau) \ne \varnothing$ for $\tau \in D$, this implies that $\mathscr{G}_{\Theta}(\Phi + \Psi') \cap \mathscr{G}_{\Theta}(\Phi + \Psi) \ne \varnothing$. Let μ be an element of this intersection, $\Lambda \in \mathscr{S}$, and $A \in \mathscr{E}^{\Lambda}$. The functions $f = \gamma_{\Lambda}^{\Phi+\Psi, \cdot}(A)$ and $f' = \gamma_{\Lambda}^{\Phi+\Psi', \cdot}(A)$ are two versions of $\mu(X_{\Lambda} \in A | \mathscr{F}_{S\setminus\Lambda})$ and thus identical μ-almost surely. By Proposition 2.24(b) of Georgii (1988), f and f' are uniform limits of local functions. Moreover, any two Gibbs measures are mutually absolutely continuous on $\mathscr{F}_{\Delta}$ for all $\Delta \in \mathscr{S}$. Thus, if $\nu \in \mathscr{G}(\Phi + \Psi')$, then $f = f'$ ν-almost surely, whence f is a version of $\nu(X_{\Lambda} \in A | \mathscr{F}_{S\setminus\Lambda})$. This shows that $\mathscr{G}(\Phi + \Psi') \subset \mathscr{G}(\Phi + \Psi)$, and the reverse inclusion follows by symmetry. This completes the proof. □

Consider now the set

$$D^{\Phi} = \{\varphi(\nu) : \nu \in \mathscr{G}_{\Theta}(\Phi + \Psi) \text{ for some } \Psi \in V\}.$$

In view of the above, D^{Φ} coincides with the set of all negative tangents to P^{Φ} and also with the set of points where J^{Φ} admits a tangent. This gives the following lemma.

LEMMA 4.10. *For all $\Phi \in \mathscr{B}_{\Theta}$, D^{Φ} is dense in D relative to the operator norm on V^*. If V is finite dimensional, D^{Φ} contains the relative interior D^{ri} of D.*

PROOF. By (2.9), D coincides with the set of all τ for which $-\tau$ is P^{Φ}-bounded, in that $-\tau \le P^{\Phi} + c$ for some constant $c > 0$. The first assertion thus follows from a general theorem of Bishop and Phelps; compare Proposition (16.7) of Georgii (1988). The second assertion follows from the fact that a convex function on a finite dimensional space admits a tangent at every point in the relative interior of its effective domain [cf. Rockafellar (1970), page 217]. □

Lemma 4.9 also implies Theorem 2.4, as we will now show.

PROOF OF THEOREM 2.4. We will apply Theorem 1.6 to the asymptotic empirical functional $\{(F_\Lambda), F\}$ defined by

$$F = \begin{cases} \langle \cdot , \Phi \rangle, & \text{if } \varphi(\nu) \in K, \\ \infty, & \text{otherwise} \end{cases}$$

and

$$F_\Lambda = \begin{cases} H_\Lambda^{\Phi^\Lambda, \mathrm{per}}, & \text{if } \tau_{\Lambda, V} \in K, \\ \infty, & \text{otherwise.} \end{cases}$$

The measures μ_Λ^F in (1.12) then coincide with the microcanonical distributions $\gamma_\Lambda^{\Phi^\Lambda, \mathrm{per}}(\cdot | \tau_{\Lambda, V} \in K)$. Hypothesis (1.14) follows from the assumption $K^0 \cap D \neq \varnothing$ and the convexity of K (cf. the remark after Theorem 1.2). We thus only need to show that $M^F \subset \mathscr{G}_\Theta(\Phi + \Psi)$ for some $\Psi \in V$. But, clearly,

$$M^F \subset \bigcup \{\mathscr{C}^\Phi(\tau) : \tau \in K_{\min}^\Phi\}.$$

Hence, if $K_{\min}^\Phi$ is a singleton the result follows immediately from Lemma 4.9. In the general case we pick any $\tau_0 \in K_{\min}^\Phi \subset D^\Phi$. Lemma 4.9 provides us with some $\Psi \in V$ such that $\mathscr{C}^\Phi(\tau_0) \subset \mathscr{G}_\Theta(\Phi + \Psi)$. For any other $\tau \in K_{\min}^\Phi$ we consider an arbitrary convex combination $\tau' = s\tau + (1 - s)\tau_0$, $0 < s < 1$. Since K and J^Φ are convex, $\tau' \in K_{\min}^\Phi$ and thus $\tau' \in D^\Phi$. Let $-\Psi' \in \partial J^\Phi(\tau')$. Then for all $\sigma, \sigma_0 \in V^*$ we have, setting $\sigma' = s\sigma + (1 - s)\sigma_0$,

$$\begin{aligned} sJ^\Phi(\tau + \sigma) &+ (1 - s)J^\Phi(\tau_0 + \sigma_0) \\ &\geq J^\Phi(\tau' + \sigma') \geq J^\Phi(\tau') - \sigma'(\Psi') \\ &= s\big[J^\Phi(\tau) - \sigma(\Psi')\big] + (1 - s)\big[J^\Phi(\tau_0) - \sigma_0(\Psi')\big] \end{aligned}$$

because J^Φ is constant on $K_{\min}^\Phi$. Setting either $\sigma = 0$ or $\sigma_0 = 0$ we see that $-\Psi' \in \partial J^\Phi(\tau) \cap \partial J^\Phi(\tau_0)$. Lemma 4.9 thus shows that

$$\mathscr{C}^\Phi(\tau) \cup \mathscr{C}^\Phi(\tau_0) \subset \mathscr{G}_\Theta(\Phi + \Psi'),$$

and its last sentence implies further that $\mathscr{G}_\Theta(\Phi + \Psi') = \mathscr{G}_\Theta(\Phi + \Psi)$. Hence $\mathscr{C}^\Phi(\tau) \subset \mathscr{G}_\Theta(\Phi + \Psi)$ for all $\tau \in K_{\min}^\Phi$, and the proof is complete. □

4.4. *Approximation of Gibbs measures by microcanonical distributions.* This section is devoted to the proof of Theorem 3.1. We begin with a lemma of general interest which refers to the general setting of subsection 3.1. That is, we fix any $\Phi \in \mathscr{B}_\Theta$ and a vector $\mathbf{\Psi} = (\Psi^1, \dots, \Psi^N)$ of potentials and consider the function $J^\Phi = J_{\mathbf{\Psi}}^\Phi$ in (3.2) and its effective domain $D = D_{\mathbf{\Psi}}$ defined by (3.3).

LEMMA 4.11. *In the setting described above, the following statements are equivalent.*

(i) *D is not contained in any hyperplane.*
(ii) *J^{Φ} is essentially smooth, that is, $D^0 \neq \varnothing$ and J^{Φ} is differentiable on D^0 with $|\operatorname{grad} J^{\Phi}(x)| \to \infty$ as $D^0 \ni x \to \partial D$.*
(iii) *The function p: $t \to P(\Phi + t \cdot \mathbf{\Psi})$ on $\mathbb{R}^N$ is strictly convex.*
(iv) *The sets $\mathscr{G}_{\Theta}(\Phi + t \cdot \mathbf{\Psi})$, $t \in \mathbb{R}^N$, are pairwise disjoint.*

PROOF. (i) $\Rightarrow$ (iv) Suppose there exist two distinct $s, t \in \mathbb{R}^N$ such that $\mathscr{G}(\Phi + s \cdot \mathbf{\Psi}) \cap \mathscr{G}(\Phi + t \cdot \mathbf{\Psi}) \neq \varnothing$. It then follows that for each cube Λ there exists an $\omega^{\Lambda} \in \Omega$ such that $(t - s) \cdot H_{\Lambda}^{\mathbf{\Psi}, \omega^{\Lambda}}$ is constant λ^{Λ}-almost surely; compare the proof of the implication (iv) $\Rightarrow$ (i) of Theorem (2.34) in Georgii (1988). Since $\nu_{\Lambda} \ll \lambda^{\Lambda}$ for all $\nu \in \mathscr{P}_{\Theta}$ with $I(\nu) < \infty$, we see that $(t - s) \cdot \nu_{\Lambda}(H_{\Lambda}^{\mathbf{\Psi}, \omega^{\Lambda}})$ takes the same value for all these ν. Hence $(t - s) \cdot \langle \cdot, \mathbf{\Psi} \rangle$ is constant on $\{I < \infty\}$, and this means that D is contained in a hyperplane.

(iv) $\Rightarrow$ (iii) Suppose p is affine on a nondegenerate interval $[s, t] \subset \mathbb{R}^N$, and let $u = (s + t)/2$. Then

$$I^{\Phi + u \cdot \mathbf{\Psi}} = \tfrac{1}{2} I^{\Phi + s \cdot \mathbf{\Psi}} + \tfrac{1}{2} I^{\Phi + t \cdot \mathbf{\Psi}}.$$

By the variational principle stated in Remark 2.2, this implies that

$$\varnothing \neq \mathscr{G}_{\Theta}(\Phi + u \cdot \mathbf{\Psi}) \subset \mathscr{G}_{\Theta}(\Phi + s \cdot \mathbf{\Psi}) \cap \mathscr{G}_{\Theta}(\Phi + t \cdot \mathbf{\Psi}),$$

in contradiction to (iv).

(iii) $\Rightarrow$ (ii) Up to trivial transformations, J^{Φ} is the convex conjugate of p. The strict convexity of p thus implies that J^{Φ} is essentially smooth; compare Rockafellar (1970), page 253.

(ii) $\Rightarrow$ (i) This is trivial. □

We now confine ourselves to the particular case when $\Phi = 0$, $N = 1$ and thus $\mathbf{\Psi} = \Psi$ for some $\Psi \in \mathscr{B}_{\Theta}$. We set $z = \langle \lambda^S, \Psi \rangle$. Since I attains its minimum 0 at λ^S only, a glance at (2.9) shows that z is the unique point where $J = J_{\Psi}^{0}$ attains its minimum 0. By convexity, this implies that J is strictly decreasing on $]-\infty, z] \cap D$.

LEMMA 4.12. *Suppose $\mathscr{G}_{\Theta}(\Psi)$ contains a measure $\mu \neq \lambda^S$. Then $\langle \mu, \Psi \rangle \in D^0$ and $\langle \mu, \Psi \rangle < z$.*

PROOF. By Lemma 4.11, p is strictly convex. For otherwise D is a singleton (containing z), and (2.2) implies that $I^{\Psi} = I$ and thus $\mathscr{G}_{\Theta}(\Psi) = \{\lambda^S\}$; compare Remark 2.2. Hence, the left and right derivatives p'_- and p'_+ of p exist, coincide for all except at most countably many points, and are strictly increasing. Using Theorem 16.14 and Remark (16.6) of Georgii (1988), we thus obtain that for any $\nu \in \mathscr{G}_\Theta(2\Psi)$,

(4.12) $-z \le p'_+(0) < p'_-(1) \le -\langle \mu, \Psi \rangle \le p'_+(1) < p'_-(2) \le -\langle \nu, \Psi \rangle.$

As $z \in D$ and $\langle \nu, \Psi \rangle \in D$, the lemma follows. □

We are now ready for the following:

Proof of Theorem 3.1. Let $\mu \in \mathscr{G}_\Theta(\Psi)$ be given.

(i) Suppose first that $\mu = \lambda^S$. Lemma 4.11 then shows that $D = \{z\}$, whence Theorem 2.4 implies that

$$\mu = \lim_{\Lambda \uparrow S} \lambda^\Lambda\big(\cdot \,|\langle \rho_\Lambda, \Psi \rangle \le z\big),$$

that is, the conclusion of Theorem 3.1 holds with $\Psi^k = \Psi$ and $c_k = z$. So we can assume that $\mu \ne \lambda^S$. We set $c = \langle \mu, \Psi \rangle$. By Lemma 4.12, $c \in D^0$ and $c < z$.

(ii) Let $\mathscr{A} = \{A_j \colon j \ge 1\}$ be a countable generator of $\mathscr{F}$ which consists of cylinder events and is stable under finite intersections. For each $n \ge 1$ there exists some $\Psi^n \in \mathscr{B}_\Theta$ such that $\|\Psi^n - \Psi\|_0 < 1/n$, $\mathscr{G}_\Theta(\Psi^n) = \{\mu_n\}$, $|\langle \mu_n, \Psi \rangle - c| < 1/n$ and

$$\max_{1 \le j \le n} |\mu_n(A_j) - \mu(A_j)| < 1/n. \tag{4.13}$$

This follows from a theorem of Sokal combined with an earlier result of Robinson and Ruelle; see Corollary (16.38) of Georgii (1988). We can assume without loss of generality that $\mu_n \ne \lambda^S$ for all n. For, if $\mu_n = \lambda^S$ for infinitely many n, then (4.13) shows that $\mu = \lambda^S$ on $\mathscr{A}$ and thereby on $\mathscr{F}$. We let $z_n = \langle \lambda^S, \Psi^n \rangle$ denote the unique point where $J_n = J^0_{\Psi^n}$ reaches its minimum 0, and we set $c_n = \langle \mu_n, \Psi^n \rangle$. Then

$$|c_n - c| \le \|\Psi^n - \Psi\|_0 + |\langle \mu_n, \Psi \rangle - c| < 2/n.$$

By Lemma 4.12, $c_n \in \{J_n < \infty\}^0$ and $c_n < z_n$. Thus J_n reaches its minimum over $]-\infty, c_n]$ exactly at c_n. An analogue of inequality (4.12) also shows that $\beta = 1$ is the unique number β such that $\mathscr{G}_\Theta(\beta\Psi^n)$ contains some ν with $\langle \nu, \Psi^n \rangle = c_n$. Equation (3.5) thus implies that

$$\mu_n = \lim_{\Lambda \uparrow S} \lambda^\Lambda\big(\cdot \,|\langle \rho_\Lambda, \Psi^n \rangle \le c_n\big).$$

Consequently, there exists a sequence (Λ_n) of cubes with $\Lambda_n \uparrow S$ such that the associated microcanonical distributions,

$$\nu_n = \lambda^{\Lambda_n}\big(\cdot \,|\langle \rho_{\Lambda_n}, \Psi^n \rangle \le c_n\big),$$

satisfy the inequality

$$\max_{1 \le j \le n} |\nu_n(A_j) - \mu_n(A_j)| < 1/n.$$

Combining this with (4.13) we see that each accumulation point of (ν_n) equals

μ on $\mathscr{A}$ and thus on $\mathscr{F}$. In view of Lemma 4.6 and an obvious extension of (4.10) in the spirit of Remark 1.4, (ν_n) is relatively compact. Hence $\nu_n \to \nu$. $\square$

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REFERENCES

BAHADUR, R. R. and ZABELL, S. L. (1979). Large deviations of the sample mean in general vector spaces. *Ann. Probab.* **7** 587–621.

BEN AROUS, G. and BRUNAUD, M. (1990). Méthode de Laplace: Etude variationelle des fluctuations de diffusions de type "champ moyens." *Stochastics Stochastics Rep.* **31** 79–144.

BOLTHAUSEN, E. (1987). Markov process large deviations in the τ-topology. *Stochastic Process. Appl.* **25** 95–108.

BOLTHAUSEN, E. and SCHMOCK, U. (1989). On the maximum entropy principle for uniformly ergodic Markov chains. *Stochastic Process. Appl.* **33** 1–27.

CHOQUET, G. (1969). *Lectures on Analysis I, II.* Benjamin, London.

COMETS, F. (1986). Grandes déviations pour des champs de Gibbs sur Z^d. *C. R. Acad. Sci. Paris Sér. I* **303** 511-513.

CSISZÁR, I. (1984). Sanov property, generalized I-projection and a conditional limit theorem. *Ann. Probab.* **12** 768–793.

CSISZÁR, I., COVER, T. M. and CHOI, B. S. (1987). Conditional limit theorems under Markov conditioning. *IEEE Trans. Inform. Theory* **33** 788–801.

DEUSCHEL, J.-D. and STROOCK, D. W. (1989). *Large Deviations.* Academic, Boston.

DEUSCHEL, J.-D., STROOCK, D. W. and ZESSIN, H. (1991). Microcanonical distributions for lattice gases. *Comm. Math. Phys.* **139** 83–101.

DONSKER, M. D. and VARADHAN, S. R. S. (1975). Asymptotic evaluation of certain Markov process expectations for large time, I. *Comm. Pure Appl. Math.* **28** 1–47.

DONSKER, M. D. and VARADHAN, S. R. S. (1976). Asymptotic evaluation of certain Markov process expectations for large time, III. *Comm. Pure Appl. Math.* **29** 389–461.

EISELE, TH. and ELLIS, R. S. (1988). Multiple phase transitions in the generalized Curie-Weiss model. *J. Statist. Phys.* **52** 161–202.

ELLIS, R. S. (1985). *Entropy, Large Deviations, and Statistical Mechanics.* Springer, New York.

ELLIS, R. S. and NEWMAN, CH. M. (1978). The statistics of the Curie Weiss model. *J. Statist. Phys.* **19** 149–161.

ELLIS, R. S. and WANG, K. (1990). Limit theorems for the empirical vector of the Curie-Weiss-Potts model. *Stochastic Process. Appl.* **35** 59–79.

FÖLLMER, H. (1988). Random fields and diffusion processes. *Ecole d'Eté de Probabilités de Saint-Flour XV–XVII. Lecture Notes in Math.* **1362** 101–203. Springer, Berlin.

FÖLLMER, H. and OREY, S. (1988). Large deviations for the empirical field of a Gibbs measure. *Ann. Probab.* **16** 961–977.

GEORGII, H. O. (1988). *Gibbs Measures and Phase Transitions.* de Gruyter, Berlin.

GEORGII, H. O. and ZESSIN, H. (1993). Large deviations and maximum entropy principle for marked point random fields. *Probab. Theory Related Fields.* To appear.

GROENEBOOM, P., OOSTERHOFF, J. and RUYMGAART, F. H. (1979). Large deviation theorems for empirical probability measures. *Ann. Probab.* **7** 553–586.

JANŽURA, M. (1985). Large deviations for Gibbs random fields. In *5th Pannonian Symp. on Math. Statist., Visegrád, Hungary* (W. Grossmann, J. Mogyorodi, I. Vincz, and W. Wertz, eds.) 97–112.

LANFORD, O. E. (1973). Entropy and equilibrium states in classical statistical mechanics. *Statistical Mechanics and Mathematical Problems. Lecture Notes in Phys.* **20** 1–113. Springer, Berlin.

MESSER, J. and SPOHN, M. (1982). Statistical mechanics of the isothermal Lane-Emden equation. *J. Statist. Phys.* **29** 561–577.

NGUYEN, X. X. and ZESSIN, H. (1979). Ergodic theorems for spatial processes. *Z. Wahrsch. Verw. Gebiete* **48** 133–158.

OLLA, S. (1988). Large deviations for Gibbs random fields. *Probab. Theory Related Fields* **77** 343–357.

OREY, S. (1986). Large deviations in ergodic theory. In *Seminar on Stochastic Processes 1984* (E. Çinlar, K. L. Chung and R. K. Getoor, eds.) 195–249. Birkhäuser, Boston.

OREY, S. (1988). Large deviations for the empirical field of Curie-Weiss models. *Stochastics Stochastics Rep.* **25** 3–14.

PARTHASARATHY, K. R. (1961). On the category of ergodic measures. *Illinois J. Math.* **5** 648–656.

PHELPS, R. R. (1966). *Lectures on Choquet's Theorem*. Van Nostrand, New York.

PIRLOT, M. (1985). Variational principle, conjugate convex functions and "the equivalence of ensembles." *Ann. Sci. Univ. Clermont-Ferrand II. Probab. Appl.* **4** 57–68.

ROCKAFELLAR, R. T. (1970). *Convex Analysis*. Princeton Univ. Press.

TEMPELMAN, A. A. (1984). Specific characteristics and variational principle for homogeneous random fields. *Z. Wahrsch. Verw. Gebiete* **65** 341–365.

VARADHAN, S. R. S. (1966). Asymptotic probabilities and differential equations. *Comm. Pure Appl. Math.* **19** 261–286.

VARADHAN, S. R. S. (1988). Large deviations and applications. *Ecole d Eté de Probabilités de Saint-Flour XV–XVII. Lecture Notes in Math.* **1362** 1–49. Springer, Berlin.

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