Translation invariance and continuous symmetries in two-dimensional continuum systems

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Abstract: We reconsider the problem of absence of continuous symmetry breaking for systems of point particles in $\mathbf{R}^2$. Assuming smoothness and suitable decay of the interaction, we show that each tempered Gibbs measure is invariant under translations. If the particles exhibit internal degrees of freedom with continuous symmetries, the latter are also preserved. The proof is elementary and avoids the use of superstability estimates.

Keywords: classical continuous system, Gibbs measure, superstable, symmetry breaking, Mermin-Wagner theorem, Widom-Rowlinson model.

1 Introduction

In the last years there has been some progress in establishing the existence of phase transitions for systems of point particles in Euclidean space. A few recent references are the following. Chayes, Chayes and Kotecký [2] and Georgii and Häggström [6] used a random cluster representation and stochastic comparison arguments to establish phase transition in the classical two-species model of Widom and Rowlinson [17] with hard-core exclusion between unlike particles, and in a q-species "continuum Potts" model with soft interspecies repulsion and type-independent molecular background interaction. In the single-species case, Johansson [10] proved a phase transition in a one-dimensional model with slowly decaying pair interaction. More recently, Lebowitz, Mazel and Presutti [11] succeeded in showing phase transition in two-dimensional models with long but finite range interactions using perturbation about a mean field van der Waals limit. In all these cases, the different Gibbs measures constructed are translation invariant. This leads us to the question of whether translation invariance necessarily holds, i.e., under what conditions all Gibbs measures are translation invariant. (Since this is trivially the case when the Gibbs measure is unique, the interest in this question is increased by the above examples of non-uniqueness.)

For point particle systems, translation is a continuous symmetry. So our question leads us into the realm of Mermin-Wagner resp. Dobrushin-Shlosman theory of conservation of continuous symmetries in two-dimensional systems. This theory was already applied to translation invariance of continuum systems by Fröhlich and Pfister [4], and Gruber and Martin [9]. Particles with spins and continuous spin-symmetries were considered by Romerio [14] (in the presence of hard-cores), and Shlosman [15]. Whereas [9] used an unverified assumption of exponential clustering, [4] and [15] relied on the superstability estimates of Ruelle [16].

In this note we show how one can avoid the use of superstability estimates, using a variant of Pfister's argument in [13]. Our first result, Theorem 1, states that if the particles interact through a translation invariant pair interaction satisfying suitable smoothness and decay conditions then, in two spatial dimensions, any tempered Gibbs measure is translation invariant. We will also discuss the extension of this result to many-body interactions. Since general (and useful) sufficient conditions in this case are somewhat tedious to obtain, we will confine ourselves to two specific examples: soft single-species Widom-Rowlinson potentials, and m-body potentials of convolution type.

Theorem 1 can easily be extended to particles with spins or other internal degrees of freedom. If the internal degrees of freedom admit some continuous symmetries then, under natural conditions on the interaction, these internal symmetries are also preserved. This second result, Theorem 2, can be proved in essentially the same way.

Although our conditions for Theorem 2 include singular and hard-core potentials, our argument for Theorem 1 is limited to smooth potentials. Also, we do not deal here with rotation invariance. Technically, the main difference is that, for rotations, particles far from the origin are moved with arbitrarily high speed. In fact, there are some arguments indicating a possible breaking of rotation invariance in two dimensional hard-core systems, cf. [12]. Under strong clustering assumptions, rotation invariance of Gibbs measures has been established in [4, 9].

2 Set-up and results

2.1 Translation invariance

We consider point particles in the Euclidean plane $\mathbf{R}^2$. A configuration of particles is described by a subset $X \subset \mathbf{R}^2$ which is locally finite, in that $\#X \cap \Lambda < \infty$ for any bounded $\Lambda \subset \mathbf{R}^2$. We write $\mathcal{X}$ for the set of all such configurations. For any Borel set $\Lambda \subset \mathbf{R}^2$, we let $X_\Lambda = X \cap \Lambda$ be the restriction of a configuration X to Λ, and $\mathcal{X}_\Lambda = \{X \in \mathcal{X} : X \subset \Lambda\}$ the set of all configurations in Λ.

The configuration space $\mathcal{X}$ is equipped with the σ-algebra $\mathcal{F}$ generated by the counting variables $N_\Lambda(X) = \#X_\Lambda$, where Λ runs through the bounded Borel sets in $\mathbf{R}^2$. If $\Lambda \subset \mathbf{R}^2$ then $\mathcal{F}_\Lambda$ stands for the σ-algebra on $\mathcal{X}$ generated by the restriction mapping $X \to X_\Lambda$ from $\mathcal{X}$ to $\mathcal{X}_\Lambda \subset \mathcal{X}$. As usually, the reference measure on the configuration space is the Poisson point random field Q on $(\mathcal{X}, \mathcal{F})$ for some fixed activity $z > 0$. Its projection Q_Λ to $\mathcal{X}_\Lambda$, for any bounded Borel set $\Lambda \subset \mathbf{R}^2$ of area $|\Lambda|$, is determined by the well-known formula

$$\int f \, dQ_\Lambda = e^{-z|\Lambda|} \sum_{k \geq 0} \frac{z^k}{k!} \int_{\Lambda^k} dx_1 \dots dx_k \; f(\{x_1, \dots, x_k\})$$

which holds for bounded measurable functions $f : \mathcal{X}_\Lambda \to \mathbf{R}$.

Next, we introduce the translation group $(\vartheta_x)_{x \in \mathbf{R}^2}$ acting on $\mathcal{X}$. For any $x \in \mathbf{R}^2$ and $X \in \mathcal{X}$, the translate $\vartheta_x X$ of X by x is defined by $\vartheta_x X = \{y - x : y \in X\}$. It is evident that the mapping $(x, X) \to \vartheta_x X$ is measurable.

Finally, we need to introduce the concept of temperedness. We divide the plane $\mathbf{R}^2$ into quadratic cells $C_i = i + [-\frac{1}{2}, \frac{1}{2}[^2$, $i \in \mathbf{L} \equiv \mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$, and consider the particle numbers $N_i \equiv N_{C_i}$ in these cells. For any integer $n \geq 1$, we further consider the box $\Lambda_n = [-n, n[^2$

of area $v_n = 4n^2$, and we define the mean square cell particle number

$$\mathbf{s}_n(X) = \frac{1}{v_n} \sum_{i \in \Lambda_n \cap \mathbf{L}} N_i(X)^2 \tag{1}$$

in Λ_n of any $X \in \mathcal{X}$. According to Ruelle [16], a configuration $X \in \mathcal{X}$ is called *tempered* if

$$\mathbf{s}^*(X) \equiv \limsup_{n\to\infty} \ \mathbf{s}_n(X) < \infty \ . \tag{2}$$

We write $\mathcal{X}^*$ for the set of all tempered configurations. Note that $\mathcal{X}^*$ belongs to the tail σ-algebra $\ \mathcal{T} = \bigcap\{\mathcal{F}_{\Lambda^c} : \Lambda \subset \mathbf{R}^2 \text{ bounded Borel}\}$.

Our next step is to introduce the interaction between the particles. A *translation invariant pair potential* is an even measurable mapping $\varphi : \mathbf{R}^2 \to \mathbf{R}$ such that the following holds: For any bounded Borel set $\Lambda \subset \mathbf{R}^2$ and $X \in \mathcal{X}^*$, the Hamiltonian

$$H_\Lambda(X) = \sum_{\{x,y\}\subset X : \{x,y\}\cap\Lambda\neq\emptyset} \varphi(x-y) \tag{3}$$

of the configuration X_Λ in Λ with boundary condition X_{Λ^c} is well-defined (as the limit of the partial sums running over $\{x,y\} \subset \Delta$ as $\Delta \uparrow \mathbf{R}^2$ through the net of bounded Borel sets), and the partition function

$$Z_{\Lambda|X_{\Lambda^c}} = \int \exp[-H_\Lambda(X)]\, Q_\Lambda(dX_\Lambda) \tag{4}$$

is positive and finite. (In particular, φ vanishes at infinity.) It is well-known that these conditions are satisfied whenever φ is stable and lower regular in the sense of [16].

For any such potential φ and any $X \in \mathcal{X}^*$ we can define the *Gibbs distribution*

$$G_{\Lambda|X_{\Lambda^c}}(dX_\Lambda) = Z^{-1}_{\Lambda|X_{\Lambda^c}} \exp[-H_\Lambda(X)]\, Q_\Lambda(dX_\Lambda) \tag{5}$$

in Λ with boundary condition X_{Λ^c}.

Definition. A probability measure P on $(\mathcal{X}, \mathcal{F})$ is called a *tempered Gibbs measure* for φ (and z) if $P(\mathcal{X}^*) = 1$ and

$$P(A|\mathcal{F}_\Lambda) = G_{\Lambda|\cdot}(A) \quad P_{\Lambda^c}\text{-almost everywhere}$$

for any bounded Borel set $\Lambda \subset \mathbf{R}^2$ and $A \in \mathcal{F}_\Lambda$.

Sufficient conditions for the existence of tempered Gibbs measure are given in [3, 16]. These include superstability or some closely related property of the interaction. In this note, however, we take the point of view that some way we are given a tempered Gibbs measure, and ask whether it must be invariant under translations. The point is that this can be decided without any use of superstability. Instead, we need some smoothness and decay properties of φ which we state now.

Assumption A. φ is a C^2–function. Its gradient $\nabla\varphi$ satisfies

$$|\nabla\varphi(x)| \to 0 \ \text{ as } \ |x| \to \infty \ , \tag{6}$$

and its Hessian $Hess\,\varphi$ obeys the estimate

$$\|Hess\,\varphi(x)\|\,|x|^2 \leq \psi(|x|) \text{ for all } x \in \mathbf{R}^2\,, \tag{7}$$

where $\|\cdot\|$ is the operator norm of a matrix and $\psi : [0,\infty[\to[0,\infty[$ a decreasing function such that $\int_0^\infty \psi(r) r\,dr < \infty$.

Here is the first result of this note.

THEOREM 1. *Under Assumption A on the translation invariant pair potential φ, each tempered Gibbs measure P for φ is invariant under the translation group $(\vartheta_x)_{x\in\mathbf{R}^2}$, i.e., $P\circ\vartheta_x = P$ for all $x \in \mathbf{R}^2$.*

Theorem 1 can obviously be extended to the case of particles with internal degrees of freedom, which will be considered below in the context of internal symmetries. It can also be extended to many-body interactions. We defrain from stating general sufficient conditions in this case and rather treat two specific examples of many-body potentials.

EXAMPLE 1. *Smooth versions of the single-type Widom-Rowlinson model.* For a given measurable function $u : \mathbf{R}^2 \to [0,\infty]$ with $\int u(x)\wedge 1\,dx < \infty$ and any bounded Borel set $\Lambda \subset \mathbf{R}^2$ we consider the Hamiltonian

$$H_\Lambda(X) = \int \Big(1 - \exp\big[-\sum_{x\in X_\Lambda} u(x-y)\big]\Big) \exp\big[-\sum_{x\in X_{\Lambda^c}} u(x-y)\big]\,dy \tag{8}$$

which obviously exists for any $X \in \mathcal{X}$. In the special case when $u(x) = \infty$ for $|x| \leq \delta$ and $u(x) = 0$ otherwise, this Hamiltonian was introduced by [17] as the single-species marginal of a two-species model with hard-core exclusion u between particles of different type. The expression in (8) is then equal to the area of the union of δ-discs centered at the points of X_Λ, decreased by the union of δ-discs around the points of X_{Λ^c}. Therefore this case is also referred to as the area-interaction model [1]. The general case corresponds to a (soft) interspecies repulsion with potential u, cf. [6]. With the shorthand $f = 1 - e^{-u}$, the Hamiltonian (8) can be rewritten in the form

$$\begin{aligned} H_\Lambda(X) &= \int \Big(1 - \prod_{x\in X_\Lambda} (1 - f(x-y))\Big) \prod_{x\in X_{\Lambda^c}} (1 - f(x-y)) \\ &= \sum_{\alpha\subset X:\,\#\alpha<\infty,\,\alpha\cap\Lambda\neq\emptyset} \varphi(\alpha) \end{aligned}$$

with a translation invariant many-body interaction

$$\varphi(\alpha) = (-1)^{\#\alpha+1} \int \prod_{x\in\alpha} f(x-y)\,dy\,. \tag{9}$$

In Section 3.4 we shall prove the following variant of Theorem 1.

PROPOSITION 1. *Consider the many-body interaction (9), where $f = 1 - e^{-u}$ and $u \geq 0$ satisfies Assumption A (with u in place of φ). Then, for any $c \in \mathbf{R}$, every Gibbs measure for the potential $c\varphi$ is (tempered and) translation invariant.*

Example 2. *m-body interactions of convolution type.* For any fixed integer $m \geq 2$ we consider the m-body potential

$$\varphi(\alpha) = \int dy \prod_{x \in \alpha} f(x-y) \ , \quad \alpha \in \mathcal{X}, \ \#\alpha = m, \tag{10}$$

where $f : \mathbf{R}^2 \to \mathbf{R}$ is measurable and such that $|f(x)| \leq \psi(|x|)$ for all $x \in \mathbf{R}^2$ and some function ψ as in Assumption A. If $m = 2$ and f is even, we have $\varphi(\{0, \cdot\}) = f * f$. For $m = 4$, such potentials have been used in [11]; compare also the potential (9) above. The Hamiltonian in a bounded Borel set $\Lambda \subset \mathbf{R}^2$ is given by

$$H_\Lambda(X) = \sum_{\alpha \subset X:\, \#\alpha = m,\, \alpha \cap \Lambda \neq \emptyset} \varphi(\alpha) \ .$$

The estimates in Section 3.5 below will show that $H_\Lambda(X)$ is well-defined if X is m-tempered, in that (2) holds with power m in place of 2 in (1). Accordingly, we can define m-tempered Gibbs measures. The analogue of Theorem 1 in the present setting then reads as follows.

Proposition 2. *Suppose φ is a linear combination of finitely many potentials of the form (10), where f as above satisfies Assumption A (with f in place of φ), and let m^* be the maximal m occurring. Then every m^*-tempered Gibbs measure for φ is translation invariant.*

2.2 Internal symmetries

Suppose now each particle carries a "mark" σ describing its type, a spin, or some other characteristic feature. σ can be taken from an arbitrary Polish space E, equipped with a σ-finite reference measure ϱ. A configuration is then described by a set $X \subset \mathbf{R}^2 \times E$ having finitely many points in $\Lambda \times E$ for each bounded $\Lambda \subset \mathbf{R}^2$. The notations $\mathcal{X}$ and $\mathcal{X}_\Lambda$ thus get an obvious new meaning. The σ-algebra $\mathcal{F}$ on $\mathcal{X}$ is defined by the counting variables $X \to \#X \cap (\Lambda \times B)$, where Λ is a bounded Borel set in $\mathbf{R}^2$ and B any Borel set in E. The (σ-finite) reference measure Q_Λ on $\mathcal{X}_\Lambda$ is defined by the formula

$$\int f \, dQ_\Lambda = \sum_{k \geq 0} \frac{z^k}{k!} \int_{\Lambda^k} dx_1 \dots dx_k \int_{E^k} \varrho(d\sigma_1) \dots \varrho(d\sigma_k) \ f(\{(x_1, \sigma_1), \dots, (x_k, \sigma_k)\})$$

for bounded measurable $f : \mathcal{X}_\Lambda \to \mathbf{R}$. Tempered configurations are defined as before.

In the following we can also deal with hard-core particles. In this case we describe their shape by a disc $\{\nu \leq \delta/2\}$ for some norm ν on $\mathbf{R}^2$, and the notation $\mathcal{X}^*$ stands for the set

$$\{X \in \mathcal{X} : \nu(x-y) > \delta \ \text{ whenever } \ (x, \sigma), (y, \tau) \in X\}$$

of admissible configurations, rather than the set of tempered configurations. We also introduce the set

$$D = \{(x, \sigma; y, \tau) \in (\mathbf{R}^2 \times E) \times (\mathbf{R}^2 \times E) : \nu(x-y) > \delta\} \ .$$

In the case of no hard core we set $\delta = 0$. With these conventions we define a (not necessarily translation invariant) pair interaction as follows.

A *pair potential* is a symmetric measurable mapping $\varphi : D \to \mathbf{R}$ such that, for any bounded Borel set $\Lambda \subset \mathbf{R}^2$ and $X \in \mathcal{X}^*$, the Hamiltonian

$$H_\Lambda(X) = \sum_{\{(x,\sigma),(y,\tau)\}\subset X:\{x,y\}\cap\Lambda\neq\emptyset} \varphi(x,\sigma;y,\tau)$$

is well-defined, and the partition function (4) is positive and finite. Gibbs distributions and tempered Gibbs measures are then defined as before.

We consider the situation when the internal degrees of freedom admit some continuous symmetries. So, suppose $(S_t)_{t\in\mathbf{R}}$ is a one-parameter family of ϱ-preserving transformations of E such that $S_0 = id$ and $S_s \circ S_t = S_{s+t}$ for all $s, t \in \mathbf{R}$. We do not require that the mapping $t \to S_t$ is bijective. We thus might have an action of the circle group, or the action generated by an arbitrary element of any connected Lie group such as SO(N). $(S_t)_{t\in\mathbf{R}}$ also acts on configurations $X \in \mathcal{X}$ via

$$S_tX = \{(x, S_t\sigma) : (x,\sigma) \in X\} \ .$$

Here are our symmetry and smoothness assumptions on φ.

Assumption B. For all $(x,\sigma;y,\tau) \in D$, $\varphi(x,S_t\sigma;y,S_t\tau) = \varphi(x,\sigma;y,\tau)$ for all $t \in \mathbf{R}$, and the function $t \to \varphi(x,\sigma;y,S_t\tau)$ is C^2 with

$$\frac{d^2}{dt^2}\,\varphi(x,\sigma;y,S_t\tau)\;|x-y|^2 \le \psi(|x-y|)$$

for a decreasing function $\psi : [0,\infty[\to [0,\infty[$ satisfying $\int_0^\infty \psi(r)r\,dr < \infty$.

Theorem 2. *Under Assumption B on φ, each tempered Gibbs measure P for φ is invariant under $(S_t)_{t\in\mathbf{R}}$.*

Theorem 2 applies in particular to the case when E is the unit sphere in $\mathbf{R}^N$ with surface measure ϱ, S_t is the rotation by the angle t around any given axis, and φ has the form

$$\varphi(x,\sigma;y,\tau) = J(x-y)\,\sigma\cdot\tau + \chi(x-y)\ ,$$

where $J, \chi : \mathbf{R}^2 \to \mathbf{R}$ are even measurable functions such that $|J(x)|\,|x|^2 \le \psi(|x|)$ whenever $\nu(x) > \delta$. J describes the spin coupling, and χ is a stabilizing molecular interaction. Such a "continuum Heisenberg model" may be taken as a model of a ferrofluid; see [8] for the case of Ising spins. Identifying antipodal points of E and replacing the scalar product $\sigma\cdot\tau$ by $|\sigma\cdot\tau|^2$, for example, we get a model of a nematic liquid crystal.

Theorem 2 also applies to the case when $E = \mathbf{R}$ with Lebesgue measure ϱ, $S_t\sigma = \sigma + t$, and φ is as above with $\sigma\cdot\tau$ replaced by $u(\sigma-\tau)$ for an even C^2-function $u : \mathbf{R} \to \mathbf{R}$ with bounded second derivative; we then have a continuum (harmonic or anharmonic) oscillator. In this case the following corollary applies.

Corollary. *Suppose $(S_t)_{t\in\mathbf{R}}$ is dissipative, in that there exists a bounded measurable function $f \ge 0$ on E such that $\int f\,d\varrho > 0$ and $\lim_{k\to\infty} f \circ S_{t_k} = 0$ ϱ-a.e. for some sequence (t_k) in $\mathbf{R}$. If φ satisfies Assumption B, a tempered Gibbs measure for φ cannot exist.*

Theorem 2 and its corollary are analogous to well-known results on lattice spin systems with continuous symmetries, cf. Theorem (9.20) and Corollary (9.24) of [5].

3 Proofs

We begin with the proof of Theorem 1; the remaining proofs follow in Sections 3.4 to 3.6. It is sufficient to prove invariance under translations in a fixed coordinate direction a. For definiteness we set $a = (1,0)$. The proof of Theorem 1 then proceeds in three stages. In a first step, we introduce localized versions of $(\vartheta_{ta})_{t\in\mathbf{R}}$, the translation flow in direction a. Then we use Assumption A to estimate the second-order change of energy under these localized translations in terms of the mean square cell particle numbers $\mathbf{s}_n$. Finally we use the temperedness and some general arguments to complete the proof.

3.1 Localized translations

For any integer $n \geq 1$, we shall define a flow $(T_t^{(n)})_{t\in\mathbf{R}}$ on $\mathbf{R}^2$ such that

(T1) each $T_t^{(n)}$ is area preserving;

(T2) $T_t^{(n)}x = \vartheta_{ta}x$ whenever $x \in \Lambda_{n/2}$ and $x + ta \in \Lambda_{n/2}$;

(T3) $T_t^{(n)}x = x$ for all $x \notin \Lambda_n$ and $t \in \mathbf{R}$.

To ensure (T1) we construct $(T_t^{(n)})_{t\in\mathbf{R}}$ as a Hamiltonian flow associated with some function h on $\mathbf{R}^2$.

Let $h_1, h_2 \in C^3(\mathbf{R})$ be such that $h_1 = h_2' = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$ and $h_1 = h_2 = 0$ off $]-1,1[$. For $x = (x_1, x_2) \in \mathbf{R}^2$ let $h(x) = h_1(x_1)h_2(x_2)$ and $v^{(1)}(x) = (v_1(x), v_2(x))$ the Hamiltonian C^2-vector field associated to h, i.e.,

$$\begin{aligned} v_1(x) &= \frac{\partial h}{\partial x_2}(x) &&= h_1(x_1)h_2'(x_2)\,, \\ v_2(x) &= -\frac{\partial h}{\partial x_1}(x) &&= -\,h_1'(x_1)h_2(x_2)\,. \end{aligned}$$

For the given $n \geq 1$ we consider the scaled vector field

$$v^{(n)}(x) = v^{(1)}(x/n)\,, \quad x \in \mathbf{R}^2\,.$$

The differential equation $\dot{x} = v^{(n)}(x)$ then obviously admits global solutions and defines a flow $(T_t^{(n)})_{t\in\mathbf{R}}$.

By Liouville's theorem, $(T_t^{(n)})_{t\in\mathbf{R}}$ satisfies (T1). Since $v^{(n)}(x) = a$ for $x \in \Lambda_{n/2}$ and $v^{(n)}(x) = 0$ when $x \notin \Lambda_n$ we also have (T2) and (T3). From the smoothness of $v^{(1)}$ we further conclude that there exist Lipschitz constants $L, L' < \infty$ such that

$$|v^{(n)}(x) - v^{(n)}(y)| \leq \frac{L}{n}\,|x-y| \tag{11}$$

and

$$|Dv^{(n)}(x)\,v^{(n)}(x) - Dv^{(n)}(y)\,v^{(n)}(y)| \leq \frac{L'}{n^2}\,|x-y| \tag{12}$$

for all $x, y \in \mathbf{R}^2$; here $Dv^{(n)}(x)$ stands for the functional matrix of $v^{(n)}$ at x. These two estimates will be exploited in the next step.

3.2 Estimate of energy change

Next we use Assumption A for estimating the change of energy under the flow $(T_t^{(n)})_{t\in\mathbf{R}}$ introduced above. To begin, we note that Assumption A implies in particular that

$$|\nabla\varphi(x)|\,|x| \le \psi(|x|) \ \text{ for all } \ x\in\mathbf{R}^2\ . \tag{13}$$

Indeed, for any x and $s>1$ we obtain from (7) and the monotonicity of ψ that

$$\begin{aligned}|\nabla\varphi(sx)-\nabla\varphi(x)|\,|x| &\le \int_1^s dr\ \|\mathit{Hess}\,\varphi(rx)\|\,|x|^2\\ &\le \int_1^s dr\ \psi(r|x|)\,r^{-2} \ \le\ \psi(|x|)\ ,\end{aligned}$$

so that (13) follows from (6) by letting $s\to\infty$. Incidentally, it follows in the same way that φ is regular, in that $|\varphi(x)|\le\psi(|x|)$ for all x. Together with (18) below, this shows that the Hamiltonian (3) exists for any $X\in\mathcal{X}^*$. In the following it will be convenient to stipulate that the norm $|\cdot|$ on $\mathbf{R}^2$ is the maximum norm.

The required energy estimate is stated in the following lemma. We write $H_n = H_{\Lambda_n}$ for the Hamiltonian in $\Lambda_n = [-n,n[^2$, and $T_t^{(n)}X = \{T_t^{(n)}x : x\in X\}$ for the image of a configuration $X\in\mathcal{X}$ under $T_t^{(n)}$. We also set $\mathbf{s}_n^*(X) = \sup_{k\ge n}\mathbf{s}_k(X)$.

Lemma 1 *There exists a constant $K<\infty$ such that, for any $n\ge 1$, $|t|\le n$ and $X\in\mathcal{X}^*$,*

$$\tfrac12\,H_n(T_t^{(n)}X)+\tfrac12\,H_n(T_{-t}^{(n)}X)-H_n(X)\le K\,t^2\,\mathbf{s}_n^*(X)\ . \tag{14}$$

Proof. We fix an arbitrary $n\ge1$ and write T_t and v instead of $T_t^{(n)}$ and $v^{(n)}$, respectively. In view of (3), the left-hand side of (14) is equal to

$$\sum_{\{x,y\}\subset X:\{x,y\}\cap\Lambda_n\neq\emptyset}\ \tfrac12\int_{-t}^{t}(t-|s|)\,\frac{d^2}{ds^2}\,\varphi(T_sx-T_sy)\,ds\ . \tag{15}$$

We thus need to estimate $\frac{d^2}{ds^2}\,\varphi(T_sx-T_sy)$ for any $x,y\in\mathbf{R}^2$ and $|s|\le n$. Writing $\langle\cdot,\cdot\rangle$ for the inner product we obtain

$$\frac{d}{ds}\,\varphi(T_sx-T_sy) = \langle\nabla\varphi(T_sx-T_sy), v(T_sx)-v(T_sy)\rangle$$

and

$$\begin{aligned}\frac{d^2}{ds^2}\,\varphi(T_sx-T_sy) \ &=\ \Big\langle v(T_sx)-v(T_sy), \mathit{Hess}\,\varphi(T_sx-T_sy)[v(T_sx)-v(T_sy)]\Big\rangle\\ &\quad+\Big\langle\nabla\varphi(T_sx-T_sy), Dv(T_sx)\,v(T_sx)-Dv(T_sy)\,v(T_sy)\Big\rangle\ ,\end{aligned}$$

so that

$$\begin{aligned}\Big|\frac{d^2}{ds^2}\,\varphi(T_sx-T_sy)\Big| \ &\le\ \|\mathit{Hess}\,\varphi(T_sx-T_sy)\|\,|v(T_sx)-v(T_sy)|^2\\ &\quad+|\nabla\varphi(T_sx-T_sy)|\,|Dv(T_sx)\,v(T_sx)-Dv(T_sy)\,v(T_sy)|\\ &\le\ \frac{L^2+L'}{n^2}\,\psi(|T_sx-T_sy|)\ .\end{aligned} \tag{16}$$

In the last step we used (11) and (7) together with (12) and (13).

Next we estimate $|T_s x - T_s y|$ from below. Using (11) we find for any $s > 0$

$$\begin{aligned} |T_s x - T_s y| - |x-y| &\leq \int_0^s du\, |v(T_u x) - v(T_u y)| \\ &\leq \frac{L}{n} \int_0^s du\, |T_u x - T_u y| \, , \end{aligned}$$

and therefore, by Gronwall's lemma,

$$|T_s x - T_s y| \leq |x-y|\, e^{L|s|/n} \, .$$

The same inequality holds for $s < 0$. By time reversal we obtain

$$|T_s x - T_s y| \geq |x-y|\, e^{-L|s|/n} \geq |x-y|\, e^{-L}$$

and thus

$$\psi(|T_s x - T_s y|) \leq \tilde{\psi}(|x-y|) \text{ for all } |s| \leq n \, ; \tag{17}$$

here $\tilde{\psi}(r) = \psi(re^{-L})$. Combining this with (15) and (16) we see that (14) will follow once we have shown that

$$\sum_{\{x,y\} \subset X : \{x,y\} \cap \Lambda_n \neq \emptyset} \tilde{\psi}(|x-y|) \leq c\, v_n\, \mathbf{s}_n^*(X) \tag{18}$$

for some constant $c < \infty$. To this end we consider the tiling of the plane $\mathbf{R}^2$ into the cells C_i. Let $d(C_i, C_j) = (|i-j| - 1)_+$ be the distance of C_i and C_j in the maximum norm $|\cdot|$ and $\tilde{\psi}_{i-j} = \tilde{\psi}(d(C_i, C_j))$. Then the left-hand side of (18) is not larger than

$$\sum_{i \in \mathbf{L} \cap \Lambda_n, j \in \mathbf{L}} N_i(X) N_j(X)\, \tilde{\psi}_{i-j} \, .$$

To estimate this term we use the (rough) inequality $N_i N_j \leq N_i^2 + N_j^2$. The resulting sum then splits off into two parts containing the terms N_i^2 resp. N_j^2. The first part is equal to $v_n\, \mathbf{s}_n(X)\, \|\tilde{\psi}\|$, where

$$\|\tilde{\psi}\| = \sum_{j \in \mathbf{L}} \tilde{\psi}_{i-j} < \infty$$

for arbitrary $i \in \mathbf{L}$. We claim that the second part satisfies the inequality

$$\sum_{i \in \mathbf{L} \cap \Lambda_n, j \in \mathbf{L}} N_j(X)^2\, \tilde{\psi}_{i-j} \leq v_n\, \mathbf{s}_n^*(X)\, [\psi(0) + 4\, \|\tilde{\psi}\|] \tag{19}$$

which, together with the previous estimates, implies (18) and thereby the lemma.

To prove (19) we introduce the differences $\partial\tilde{\psi}(k) = \tilde{\psi}(k-1) - \tilde{\psi}(k)$ for $k \geq 1$. Then

$$\tilde{\psi}_{i-j} = \sum_{k \geq |i-j|} \partial\tilde{\psi}(k) \text{ when } i \neq j \, . \tag{20}$$

Separating the terms with $i = j$ we thus see that the left-hand side of (19) is equal to

$$\psi(0) v_n \mathbf{s}_n(X) + \sum_{k \geq 1} \partial\tilde{\psi}(k) \sum_{j \in \mathbf{L}} N_j(X)^2\, \#\{i \in \mathbf{L} \cap \Lambda_n : |i-j| \leq k\} \, .$$

The last cardinality vanishes unless $j \in \mathbf{L} \cap \Lambda_{n+k} \subset \mathbf{L} \cap \Lambda_{2(n\vee k)}$, in which case it is at most $v_n \wedge v'_k$, where $v'_k = (2k+1)^2 \geq v_k$. The last inner sum is therefore not larger than

$$[v_n \wedge v'_k]\, v_{2(n\vee k)}\, \mathbf{s}_{2(n\vee k)}(X) \leq 4\, v_n\, v'_k\, \mathbf{s}^*_n(X) \ .$$

This proves (19) because $\sum_{k\geq 1} \partial\tilde{\psi}(k)\, v'_k = \|\tilde{\psi}\|$. The proof of Lemma 1 is therefore complete.

3.3 Comparison of probabilities

We will now use Lemma 1 to complete the proof of Theorem 1. The key observation is the following inequality for tempered Gibbs measures. As before, $a = (1,0)$. Let K be the constant of Lemma 1.

Lemma 2 *For any tempered Gibbs measure P and any $t \in \mathbf{R}$,*

$$\tfrac{1}{2}\, P \circ \vartheta_{ta} + \tfrac{1}{2}\, P \circ \vartheta_{-ta} \geq e^{-Kt^2 \mathbf{s}^*} P \ .$$

Proof. Suppose $A \in \mathcal{F}$ is local, in that it only depends on the configuration in a bounded set, and let $n \geq 1$ be so large that $|t| \leq n$ and $A \in \mathcal{F}_{\Lambda_{n/2}}$, $\vartheta_{\pm ta} A \in \mathcal{F}_{\Lambda_{n/2}}$. Then, by (T2), $\vartheta_{ta} A = T^{(n)}_t A$ and $\vartheta_{-ta} A = T^{(n)}_{-t} A$. By (T1) and (T3), $T^{(n)}_{\pm t}$ preserves the Poisson point random field Q_{Λ_n}. In view of (5) and the definition of tempered Gibbs measures, it follows that $P \circ T^{(n)}_{\pm t}$ is absolutely continuous with respect to P with Radon-Nikodym density $\exp[-H_n \circ T^{(n)}_{\pm t} + H_n]$. So we can write

$$\begin{aligned}
\tfrac{1}{2}\, P(\vartheta_{ta} A) + \tfrac{1}{2}\, P(\vartheta_{-ta} A) &= \tfrac{1}{2}\, P(T^{(n)}_t A) + \tfrac{1}{2}\, P(T^{(n)}_{-t} A) \\
&= \int_A \Big(\tfrac{1}{2}\, \exp[-H_n \circ T^{(n)}_t + H_n] + \tfrac{1}{2}\, \exp[-H_n \circ T^{(n)}_{-t} + H_n] \Big)\, dP \\
&\geq \int_A \exp\big[-\tfrac{1}{2}\, H_n \circ T^{(n)}_t - \tfrac{1}{2}\, H_n \circ T^{(n)}_{-t} + H_n \big]\, dP \\
&\geq \int_A e^{-Kt^2 \mathbf{s}^*_n}\, dP \ .
\end{aligned}$$

In the last two inequalities we used the convexity of the exponential function and Lemma 1. Letting $n \to \infty$ we obtain from Fatou's lemma

$$\tfrac{1}{2}\, P(\vartheta_{ta} A) + \tfrac{1}{2}\, P(\vartheta_{-ta} A) \geq \int_A e^{-Kt^2 \mathbf{s}^*}\, dP \ .$$

As A was an arbitrary local event, the lemma thus follows from the monotone class theorem.

Lemma 2 implies Theorem 1 as follows. Suppose P is an *extreme* tempered Gibbs measure, and let $t \in \mathbf{R}$. Since $\mathbf{s}^* < \infty$ with P-probability 1, we conclude from Lemma 2 that $P \ll \tfrac{1}{2}\, P \circ \vartheta_{ta} + \tfrac{1}{2}\, P \circ \vartheta_{-ta}$. On the other hand, $P \circ \vartheta_{ta}$ and $P \circ \vartheta_{-ta}$ are also extreme tempered Gibbs measures. Therefore, if $P \circ \vartheta_{ta} \neq P$ then also $P \circ \vartheta_{-ta} \neq P$, whence $P \circ \vartheta_{ta}$ and $P \circ \vartheta_{-ta}$ would be singular with respect to P, cf. Theorem (7.7)(d) of [5]. Hence $\tfrac{1}{2}\, P \circ \vartheta_{ta} + \tfrac{1}{2}\, P \circ \vartheta_{-ta}$ would be singular with respect to P. Since this is not the case, it follows that $P \circ \vartheta_{ta} = P$. By the extreme decomposition theorem (cf. Theorem (7.26) of [5]), the same holds for any tempered Gibbs measure, and Theorem 1 is proved.

3.4 Smooth Widom-Rowlinson potentials

In this subsection we prove Proposition 1. We start by noting that every Gibbs measure for (any multiple of) the Hamiltonian (8) is tempered. This follows from the fact that every such Gibbs measure is stochastically dominated by the Poisson point random field Q or, in the case of a negative multiple, by the Poisson point random field for a suitably chosen activity, cf. Examples 2.1 and 2.3 of [7]. In view of the cancellations due to the alternating signs in the many-body interaction φ in (9), it is preferable to deal directly with the Hamiltonian (8) rather than with the potential φ.

We consider again the localized translations $(T_t^{(n)})_{t\in\mathbf{R}}$. In view of (T1), we have for any $n \geq 1$

$$H_n(T_t^{(n)}X) = \int (1 - e^{-f(t,y)})\, e^{-g(t,y)}\; dy \;,$$

where

$$f(t,y) = \sum_{x\in X_\Lambda} u(T_t^{(n)}x - T_t^{(n)}y)$$

and

$$g(t,y) = \sum_{x\in X_{\Lambda^c}} u(T_t^{(n)}x - T_t^{(n)}y) \;.$$

Denoting partial derivatives with respect to t by a sub-t and estimating the exponentials by 1 we obtain

$$\Big|\frac{\partial^2}{\partial t^2}(1 - e^{-f})\, e^{-g}\Big| \leq |f_{tt}| + |g_{tt}| + (|f_t| + |g_t|)^2 \;.$$

In view of (T3), $g_t(t,y) = 0$ when $y \notin \Lambda_n$. Also, in analogy to (13), (16) and (17) we have for $|t| \leq n$

$$\begin{aligned}
|f_t(t,y)| &\leq \frac{L}{n} \sum_{x\in X_{\Lambda_n}} \tilde\psi(|x-y|) \;,\\
|f_{tt}(t,y)| &\leq \frac{L^2 + L'}{n^2} \sum_{x\in X_{\Lambda_n}} \tilde\psi(|x-y|) \;,
\end{aligned}$$

and similarly for g (with Λ_n^c instead of Λ_n). We define $I_n(x,y) = 0$ for $x, y \notin \Lambda_n$ and $I_n(x,y) = 1$ otherwise. Then we obtain, using the Cauchy-Schwarz inequality and (19),

$$\begin{aligned}
\Big|\frac{d^2}{dt^2} H_n(T_t^{(n)}X)\Big| &\leq \frac{L^2 + L'}{n^2} \int dy \sum_{x\in X} I_n(x,y)\, \tilde\psi(|x-y|)\\
&\quad + \frac{L^2}{n^2} \int dy\, \Big(\sum_{x\in X} I_n(x,y)\, \tilde\psi(|x-y|)\Big)^2\\
&\leq \frac{L^2 + L'}{n^2} \sum_{i\in\mathbf{L}} \sum_{j\in\mathbf{L}} N_j(X)\, I_n(i,j)\, \tilde\psi_{i-j}\\
&\quad + \frac{L^2}{n^2} \sum_{i\in\mathbf{L}} \|\tilde\psi\| \sum_{j\in\mathbf{L}} N_j(X)^2\, I_n(i,j)\, \tilde\psi_{i-j}
\end{aligned}$$

$$\begin{aligned}&\le \frac{c}{n^2} \sum_{i\in\mathbf{L}\cap\Lambda_n, j\in\mathbf{L}} N_j(X)^2\, \tilde{\psi}_{i-j}\\ &\le K\, \mathbf{s}_n^*(X)\end{aligned}$$

with suitable constants c and K. Hence, any multiple of H_n satisfies an analogue of Lemma 1, and Proposition 1 follows as in Section 3.3.

3.5 Many-body interactions of convolution type

Here we prove Proposition 2. Let φ be given by (10) with an f satisfying Assumption A. In particular, (13) holds with f instead of φ, and $|f(x)| \le \psi(|x|)$ for all x. Fixing any $n \ge 1$, we conclude from (T1) that

$$\varphi(T_t^{(n)}\alpha) = \int dy \prod_{x\in\alpha} f(T_t^{(n)}x - T_t^{(n)}y)$$

for all $\alpha \in \mathcal{X}$ with $\#\alpha = m$. We differentiate twice with respect to t and use (11) and (12) together with Assumption A to estimate the f-derivatives in terms of ψ, in analogy to (16) and (17). This gives

$$\Big|\frac{d^2}{dt^2}\,\varphi(T_t^{(n)}\alpha)\Big| \le \frac{c}{n^2} \int dy \prod_{x\in\alpha} \tilde{\psi}(|x-y|)$$

with $c = (2L^2 + L')m^2$. Hence, for any m-tempered $X \in \mathcal{X}$,

$$\begin{aligned}\Big|\frac{d^2}{dt^2}\,H(T_t^{(n)}X)\Big| &\le \frac{c}{n^2} \sum_{x_1\in X\cap\Lambda_n,\, x_2,\dots,x_m\in X} \int dy \prod_{\ell=1}^m \tilde{\psi}(|x_\ell - y|)\\ &\le \frac{c}{n^2} \sum_{i_1\in\mathbf{L}\cap\Lambda_n,\, i_2,\dots,i_m\in\mathbf{L}} N_{i_1}(X)\dots N_{i_m}(X) \sum_{k\in\mathbf{L}} \prod_{\ell=1}^m \tilde{\psi}_{i_\ell - k}\,.\end{aligned}$$

In view of the inequality $N_{i_1}\dots N_{i_m} \le N_{i_1}^m + \dots N_{i_m}^m$, we therefore only need to show that, for any $1 \le q \le m$,

$$\sum_{i_1\in\mathbf{L}\cap\Lambda_n,\, i_2,\dots,i_m\in\mathbf{L}} N_{i_q}(X)^m \sum_{k\in\mathbf{L}} \prod_{\ell=1}^m \tilde{\psi}_{i_\ell-k} \le \tilde{c}\, v_n\, \mathbf{s}_n^*(X) \tag{21}$$

with some constant $\tilde{c} < \infty$. Here $\mathbf{s}_n^* = \sup_{k\ge n} \mathbf{s}_k$, and $\mathbf{s}_k$ is defined by (1) with power m instead of 2.

Suppose first that $q = 1$. Summing first over $i_2, \dots, i_m \in \mathbf{L}$ and then over $k \in \mathbf{L}$ we find that the left-hand side of (21) is equal to

$$\|\tilde{\psi}\|^m \sum_{i\in\mathbf{L}\cap\Lambda_n} N_i(X)^m = \|\tilde{\psi}\|^m\, v_n\, \mathbf{s}_n(X)\,.$$

In the case $2 \le q \le m$ we can assume by symmetry that $q = 2$. Summing over $i_3, \dots, i_m \in \mathbf{L}$ we see that the left-hand side of (21) coincides with

$$\|\tilde{\psi}\|^{m-2} \sum_{i\in\mathbf{L}\cap\Lambda_n,\, j\in\mathbf{L}} N_j(X)^m \sum_{k\in\mathbf{L}} \tilde{\psi}_{i-k}\, \tilde{\psi}_{j-k}\,.$$

Ignoring the factor $\|\tilde{\psi}\|^{m-2}$ in front we obtain for the partial sum over all terms with $k = i$ the expression

$$\psi(0) \sum_{i \in \mathbf{L} \cap \Lambda_n,\, j \in \mathbf{L}} N_j(X)^m \, \tilde{\psi}_{j-i}$$

which, in analogy to (19), admits an upper bound as required in (21). For the partial sum over the terms with $k \neq i$ we find using (20)

$$\sum_{\ell \geq 1} \partial\tilde{\psi}(\ell) \sum_{k \in \mathbf{L} \cap \Lambda_{n+\ell},\, j \in \mathbf{L}} N_j(X)^m \, \tilde{\psi}_{j-k} \, \#\{i \in \mathbf{L} \cap \Lambda_n : |i - k| \leq \ell\} \; .$$

The last cardinality is at most $v_n \wedge v'_\ell$. By the proof of (19), the double inner sum is dominated by a constant multiple of $v_{n+\ell}\, \mathbf{s}^*_{n+\ell}(X)$, which is not larger than $4\,(v_n \vee v'_\ell)\, \mathbf{s}^*_n(X)$. The expression in the last display is therefore dominated by a multiple of

$$\sum_{\ell \geq 1} \partial\tilde{\psi}(\ell)\, v_n\, v'_\ell\, \mathbf{s}^*_n(X) = v_n \, \|\tilde{\psi}\| \, \mathbf{s}^*_n(X) \; ,$$

and the proof of (21) is complete. Consequently, any finite linear combination of potentials of the form (10) satisfies a counterpart to Lemma 1, and Proposition 2 follows as in Section 3.3.

3.6 Internal symmetries

The proof of Theorem 2 is again similar to that of Theorem 1 and, in fact, much simpler. For any $n \geq 1$, we define a localized internal symmetry group $(S^{(n)}_t)_{t \in \mathbf{R}}$ on configurations $X \subset \mathbf{R}^2 \times E$ by

$$S^{(n)}_t X = \{(x, S_{t f_n(x)}\sigma) : (x, \sigma) \in X\} \; ,$$

where $f_n(x) = 1 \wedge (2 - \frac{2}{n}|x|)_+$ for $x \in \mathbf{R}^2$. Assumption B then implies that

$$\begin{aligned} \frac{d^2}{ds^2}\, \varphi(x, S_{s f_n(x)}\sigma; y, S_{s f_n(y)}\tau) \;&\leq\; (f_n(x) - f_n(y))^2 \, |x - y|^{-2} \, \psi(|x - y|) \\ &\leq\; \frac{4}{n^2}\, \psi(|x - y|) \end{aligned}$$

whenever $(x, \sigma; y, \tau) \in D$. Just as in the proof of Lemma 1 this leads to the inequality

$$\tfrac{1}{2}\, H_n(S^{(n)}_t X) + \tfrac{1}{2}\, H_n(S^{(n)}_{-t} X) - H_n(X) \leq K\, t^2\, \mathbf{s}^*_n(X)$$

for arbitrary $n \geq 1$, $t \in \mathbf{R}$, $X \in \mathcal{X}^*$ and suitable $K < \infty$. Since Q_{Λ_n} is invariant under $(S^{(n)}_t)_{t \in \mathbf{R}}$, this implies a counterpart to Lemma 2, and Theorem 2 follows immediately.

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