Random geometric analysis of the 2d Ising model

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Plan:

Foundations

1. Gibbs measures
2. Stochastic order
3. Percolation

The Ising model

4. Random clusters and phase transition
5. 2d: only two invariant phases
6. 2d: no non-invariant phase

1. Gibbs measures

Seek: Model for spatial system
with many interacting components

Ex: Ferromagnet (iron, nickel)
consisting of many "spins" forming a crystal lattice

Ingredients:

- infinite lattice, e.g. $\mathbb{Z}^d$ (≈ large finite lattice)
- spin orientations $\in$ finite set S

spin configuration $\sigma = (\sigma_i)_{i\in\mathbb{Z}^d} \in S^{\mathbb{Z}^d} =: \Omega$

Spin dependence: prescribe conditional probabilities

$$\text{Prob}(\sigma \text{ in } \Lambda \,|\, \eta \text{ off } \Lambda) = \gamma_\Lambda(\sigma|\eta)$$

for $\Lambda \subset\subset \mathbb{Z}^d,\ \sigma \in S^\Lambda,\ \eta \in S^{\Lambda^c}$ (consistently)

- *Markovian case:* $\gamma_\Lambda(\sigma|\eta) = \gamma_\Lambda(\sigma|\eta_{\partial\Lambda})$
  for $\partial\Lambda = \{d(\cdot\,,\Lambda) = 1\}$

- *Gibbsian case:* $\gamma_\Lambda(\sigma|\eta) \propto \exp[-\beta\, H_\Lambda(\sigma\eta)]$
  for some Hamiltonian H_Λ and inv. temperature $\beta > 0$

Ex: *Ising model:* W. Lenz '20, E. Ising '24

$S = \{1, -1\}$ (up–down)

$$H_\Lambda(\xi) = \sum_{\{i,j\}\cap\Lambda\neq\emptyset,\, |i-j|=1} 1_{\{\xi_i\neq\xi_j\}}$$

– adjacent different spins are penalized

– nearest-neigbor interaction $\Rightarrow \gamma$ Markovian

Def: Dobrushin '68, Lanford–Ruelle '69

μ on Ω *Gibbs measure* for $\gamma = (\gamma_\Lambda)_{\Lambda\subset\subset\mathbb{Z}^d}$ if

$$\mu(\sigma \text{ in } \Lambda \,|\, \eta \text{ off } \Lambda) = \gamma_\Lambda(\sigma|\eta) \quad \mu\text{-a.e. } \eta$$

$$\forall\, \sigma \in S^\Lambda,\ \Lambda \subset\subset \mathbb{Z}^d$$

$\mathcal{G} \equiv \mathcal{G}(\gamma)$ set of all Gibbs measures

equilibrium states for physical system with interaction γ

Facts:

- γ (almost) Markovian $\Rightarrow \underset{\Lambda\uparrow\mathbb{Z}^d}{\text{acc}}\, \gamma_\Lambda(\cdot\,|\eta_{\Lambda^c}) \subset \mathcal{G}$

  $\Rightarrow \mathcal{G} \neq \emptyset$

- $\mathcal{G}$ convex $\rightsquigarrow$ $\text{ex}\,\mathcal{G}$ = extremal points

- $\mu \in \mathcal{G}$ extremal $\iff$ μ trivial on $\mathcal{T} = \bigcap_{\Lambda\subset\subset\mathbb{Z}^d} \mathcal{F}_{\Lambda^c}$

  "macroscopically deterministic"

- $\mu, \nu \in \text{ex}\,\mathcal{G},\ \mu \neq \nu \Rightarrow \mu \neq \nu$ on $\mathcal{T}$

  "macroscopically distinguishable"

- $\mu \in \text{ex}\,\mathcal{G} \Rightarrow \mu = \lim_{\Lambda\uparrow\mathbb{Z}^d} \gamma_\Lambda(\cdot\,|\eta_{\Lambda^c})$ for μ-a.e. η

  "finite system approximation"

- $\mu \in \mathcal{G} \Rightarrow \mu = \int_{\text{ex}\,\mathcal{G}} \nu\, w(d\nu)$ for a unique w

  "extremal decomposition"

  $\Rightarrow$ any $\mathcal{G}$-typical $\sigma \in \Omega$ is typical for some $\mu \in \text{ex}\,\mathcal{G}$

$\rightsquigarrow$ *Def:* Any $\mu \in \operatorname{ex} \mathcal{G}$ is called a *phase.*

If $|\operatorname{ex} \mathcal{G}| > 1$: **phase transition**

"macroscopic ambivalence"

Question:

What are the driving forces giving rise
to phase transition?
Is there any stochastic mechanism relating
microscopic and macrosopic behavior of spins?

Will see:

A possible such mechanism is the
formation of infinite clusters
in suitable random graphs defined by the spins.
Such infinite clusters serve as a link
between individual and collective behavior.

2. Stochastic order (FKG–order)

$S \subset \mathbb{R}$, $\mathcal{L}$ any index set $\;(= \mathbb{Z}^d\,)$

$\Rightarrow\;\; \Omega = S^{\mathcal{L}}$ partially ordered:

$$\xi \le \eta \iff \xi_i \le \eta_i \;\; \forall\, i \in \mathbb{Z}^d$$

$f : \Omega \to \mathbb{R}$ *increasing* $\iff$ $f(\xi) \le f(\eta)$ for $\xi \le \eta$

$A \subset \Omega$ *increasing* $\iff$ 1_A increasing

Def: $\mu \preceq \nu$ *"stochastically smaller"*

if $\mu(f) \le \nu(f)$ for all increasing f

Thm: Strassen '65

$$\mu \preceq \nu \iff \exists \text{ coupling } P \text{ on } \Omega \times \Omega \text{ of } \mu, \nu \quad \text{s.t. } P((\xi,\eta) : \xi \le \eta) = 1$$

Pf: "$\Leftarrow$" $\mu(f) = \int_{\{\xi \le \eta\}} P(d\xi, d\eta) \underbrace{f(\xi)}_{\le f(\eta)} \le \nu(f)$

"$\Rightarrow$" If $|\mathcal{L}| = 1$, $\Omega = S$:

$P =$ distribution of $(F_\mu^{-1}(U), F_\nu^{-1}(U))$ with $U \sim \text{Uni}(0,1)$

General case: deep via Hahn–Banach $\square$

Thm: Holley '74

$$\mu \preceq \nu \text{ whenever } |\mathcal{L}| < \infty,\ \mu, \nu > 0 \text{ and}$$
$$\mu_i(\cdot\,|\xi) \preceq \nu_i(\cdot\,|\eta) \quad \forall\, i \in \mathcal{L},\ \xi \le \eta$$

Pf: Define irreducible transition matrix M on $\Omega \times \Omega$ by

$$M_i(\xi, \eta; \cdot, \cdot) = \begin{cases} \text{coupling of } \mu_i(\cdot\,|\xi) \text{ and } \nu_i(\cdot\,|\eta) & \text{if } \xi \le \eta \\ \mu_i(\cdot\,|\xi) \times \nu_i(\cdot\,|\eta) & \text{otherwise} \end{cases}$$

and $M = |\mathcal{L}|^{-1} \sum_{i \in \mathcal{L}} M_i$.

Stationary distribution of M is Strassen coupling of μ, ν □

Cor: Fortuin, Kasteleyn, Ginibre '71

$$|\mathcal{L}| < \infty,\ \mu > 0,\ \mu_i(\cdot\,|\xi) \preceq \mu_i(\cdot\,|\eta)\ \forall\, i \in \mathcal{L},\ \xi \le \eta$$

$\Rightarrow$ μ has positive correlations, i.e.

$$\mu(fg) \ge \mu(f)\,\mu(g) \quad \forall \text{ increasing } f, g$$

Pf: W.l.o.g. $f > 0$, $\mu(f) = 1$. Define $\nu = f\,\mu$.
Then Holley's conditions hold for μ, ν □

Ex: $S = \{0, 1\}$ $\Rightarrow$ Bernoulli measure μ_p stochastically increasing in p, has positive correlations

Application to the Ising model

Lemma: $\xi \le \eta$ off $i \;\Rightarrow\; \gamma_i(\cdot\,|\xi) \preceq \gamma_i(\cdot\,|\eta)$

Pf: $\gamma_i(1|\xi) = 1\Big/\Big(1 + \exp[-\beta \sum_{j\in\partial i} \xi_j]\Big)$

is increasing in ξ $\square$

$\Rightarrow$ *Sandwich property:* $\forall \Delta \subset \Lambda \subset\subset \mathbb{Z}^d, \eta \in S^{\Lambda^c}$

$$\gamma_\Delta(\cdot\,|-) \preceq \gamma_\Lambda(\cdot\,|\eta) \preceq \gamma_\Delta(\cdot\,|+)$$

Thm: Lebowitz, Martin-Löf '72

- $\exists\, \mu^+ = \downarrow \lim_{\Lambda\uparrow\mathbb{Z}^d} \gamma_\Lambda(\cdot\,|+),\ \mu^- = \uparrow \lim_{\Lambda\uparrow\mathbb{Z}^d} \gamma_\Lambda(\cdot\,|-) \in \mathcal{G}_\Theta$
- $\forall\, \mu \in \mathcal{G} \qquad \mu^- \preceq \mu \preceq \mu^+, \quad \mu^-, \mu^+ \in \mathrm{ex}\,\mathcal{G}$
- Each $\mu \in \mathrm{ex}\,\mathcal{G}$ has positive correlations

Cor: $|\mathcal{G}| = 1 \iff \mu^+ = \mu^-$

$$\iff \mu^+(\sigma_i = 1) = \mu^-(\sigma_i = 1)\ \forall i$$

$$\iff \mu^+(\sigma_0 = 1) = \tfrac{1}{2}$$

3. Percolation

A. Bernoulli percolation

Broadbent–Hammersley '57, Flory '41:
percolation of water through porous medium

$(\xi_i)_{i\in\mathbb{Z}^d}$ i.i.d. on $\{0,1\}$ with $\mu_p(\xi_i = 1) = p$

$$\xi = \{i \in \mathbb{Z}^d : \xi_i = 1\} \qquad \text{“open sites”}$$

defines random subgraph of $\mathbb{Z}^d$

vertex set ξ, edge set $E(\xi) = \{e = \{i,j\} \subset \xi : |i-j| = 1\}$

Cluster of ξ = maximal connected subset

Qu: $\exists$ infinite cluster with prob. > 0?

- $\exists$ threshold p_c s.t.

$$\mu_p(\exists \text{ infinite cluster}) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \phantom{\text{for }} p > p_c \end{cases}$$

- $\mu_p(\exists \text{ infinite cluster}) = 1 \iff \mu_p(0 \leftrightarrow \infty) > 0$

Pf: event increasing $\Rightarrow$ probability increasing

Kolmogorov zero-one $\Rightarrow \in \{0,1\}$ $\square$

Non-triviality of the threshold:

$$d \geq 2 \;\Rightarrow\; 0 < p_c < 1$$

Pf: $\Delta \subset\subset \mathbb{Z}^d,\; n := d(0, \partial\Delta) \;\Rightarrow$

$$\mu_p(0 \leftrightarrow \infty) \leq \mu_p(0 \leftrightarrow \partial\Delta)$$

$$\leq 2d\,(2d-1)^{n-1}p^n \;\underset{n\to\infty}{\longrightarrow}\; 0 \quad \text{if } (2d-1)p < 1$$

$\Rightarrow\; p_c \geq \frac{1}{2d-1}$

Conversely: *Peierls argument* '33

W.l.o.g. $d = 2$. Fix $\Delta \subset\subset \mathbb{Z}^2$.

Consider $C_\Delta = \Delta \cup \bigcup\{\text{clusters hitting } \Delta\}$

$$\mu_p(\text{no infinite cluster hits } \Delta)$$

$$\leq \mu_p(\partial C_\Delta \text{ contains a } *\text{-circuit around } \Delta)$$

$$\leq \sum_{n \geq |\partial\Delta|} n\, 7^{n-1}(1-p)^n \;<\; 1$$

if $p > \frac{6}{7}$ and Δ is large. Hence $p_c \leq \frac{6}{7}$ $\square$

Similar:

bond percolation with i.i.d. random edges

B. Invariant percolation

$(\xi_i)_{i\in\mathbb{Z}^d}$ translation invariant $\{0,1\}$-valued

Qu: How many infinite clusters?

Thm: Burton–Keane '89

μ invariant under $p\mathbb{Z}^d$-translations s.t. $\mu(\sigma \text{ in } \Lambda \,|\, \eta \text{ off } \Lambda) > 0 \quad \forall\, \sigma, \eta, \Lambda$ *"finite energy"* $\Rightarrow\ \mu(\exists\ \le 1 \text{ infinite cluster}) = 1$

Pf: W.l.o.g. μ ergodic under $p\mathbb{Z}^d$ translations

$N := \#$ infinite clusters is invariant

$\Rightarrow\ \exists\, k \in \{0, 1, \dots, \infty\}$ s.t. $\mu(N = k) = 1$

Case $k = 2$ For large Δ

$$\mu(2 \text{ infinite clusters hit } \Delta) > 0$$

finite energy $\Rightarrow$

$$\mu(\Delta \subset \xi,\ 2 \text{ infinite clusters hit } \Delta) > 0$$

$\Rightarrow\ 2 = 1$

Case $k \geq 3$ Say i is *triple point* if $i \in \xi$ and

3 clusters of $\xi \setminus \{i\}$ hit ∂i

$A_i := \{i$ is triple point$\}$

finite energy $\Rightarrow$

$\mu(A_0) =: 2\delta > 0$

ergodic theorem $\Rightarrow$ for large Λ

$$\mu(|\Lambda|^{-1} \sum_{i \in \Lambda} 1_{A_i} \geq \delta) \geq \frac{1}{2}$$

But: # triple points in $\Lambda \leq |\partial\Lambda|$ because

$$\Rightarrow \sum_{i \in \Lambda} 1_{A_i} \leq |\partial\Lambda| \leq \delta\, |\Lambda| \quad \text{for large } \Lambda$$

$$\Rightarrow \mu(\emptyset) \geq \tfrac{1}{2}$$

Ising model:

4. Random clusters and phase transition

Qu: For which $\beta > 0$ is $|\mathcal{G}| > 1$, i.e. $\mu^+ \neq \mu^-$?

Key: RC representation of $\gamma_\Lambda(\cdot\,|+)$:

Λ cube in $\mathbb{Z}^d$,

$$E(\Lambda) = \{e = \{i,j\} : |i-j| = 1,\ e \cap \Lambda \neq \emptyset\}$$

set of edges meeting Λ

Define RC distribution ϕ_Λ on $\{0,1\}^{E(\Lambda)}$ by

$$\phi_\Lambda(\eta) \ \propto\ 2^{k(\Lambda,\eta)}\ p^{|\eta|}(1-p)^{|E(\Lambda)\setminus\eta|}$$

with $p = 1-e^{-\beta}$ and $k(\Lambda,\eta) = \#$ clusters of (Λ,η)
(all clusters hitting $\partial\Lambda$ joined into a single boundary cluster)

Lemma: η random $\sim \phi_\Lambda$,

$$\sigma = \begin{cases} +1 & \text{on the boundary cluster of } (\Lambda,\eta) \\ \pm 1 & \text{with prob. } \frac{1}{2} \text{ indep. on each other cluster} \end{cases}$$

$\Rightarrow \sigma \sim \gamma_\Lambda(\cdot\,|+)$

Pf: For each σ with $\sigma \equiv +$ off Λ: $\quad \sum_\eta \phi_\Lambda(\eta)\,\pi_\Lambda(\sigma|\eta)$

$$\propto \sum_{\eta:\ \eta(i,j)=0 \text{ if } \sigma_i\neq\sigma_j} 2^{k(\Lambda,\eta)}\ p^{|\eta|}(1-p)^{|E(\Lambda)\setminus\eta|}\ 2^{-k(\Lambda,\eta)+1}$$

$$\propto (1-p)^{|\{\{i,j\}:\sigma_i\neq\sigma_j\}|} \ \propto\ \gamma_\Lambda(\sigma|+) \qquad \square$$

Thm: *phase transition* $\Longleftrightarrow$ *percolation*

$$\gamma_\Lambda(\sigma_0 = +1|+) \;=\; \tfrac{1}{2} + \tfrac{1}{2}\,\phi_\Lambda(0 \leftrightarrow \partial\Lambda)$$

Hence $|\mathcal{G}| > 1 \;\Longleftrightarrow\; \theta := \lim_{\Lambda\uparrow\mathbb{Z}^2} \phi_\Lambda(0 \leftrightarrow \partial\Lambda) > 0$

Holley $\Rightarrow$

- ϕ_Λ **stochastically increasing in** $p = 1 - e^{-\beta}$
- $\phi_\Lambda \preceq \mu_p$, **whence** $\theta \leq \mu_p(0 \leftrightarrow \infty) = 0$ **for** $p \approx 0$
- $\phi_\Lambda \succeq \mu_{\frac{p}{2-p}}$, **whence** $\theta \geq \mu_{\frac{p}{2-p}}(0 \leftrightarrow \infty) > 0$ **for** $p \approx 1$ **and** $d \geq 2$

Aizenman, Chayes, Chayes, Newman ’88 $\Rightarrow$

Thm: *Phase transition for* $d \geq 2$

$$\exists\, 0 < \beta_c < \infty \text{ s.t. } \quad |\mathcal{G}| \begin{cases} = 1 & \text{for } \beta < \beta_c \\ > 1 & \text{for } \beta > \beta_c \end{cases}$$

Qu: *How many phases for* $\beta > \beta_c$*?*

- For $d \geq 3$: infinitely many (Dobrushin ’72)
- For $d = 2$: only two

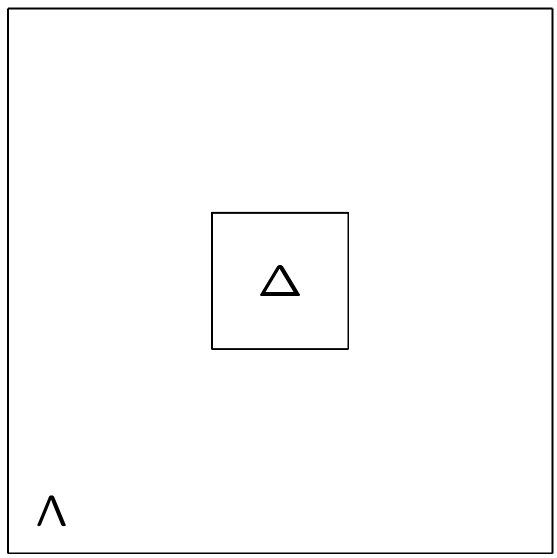
Thm: Russo, Aizenman, Higuchi ’79 / ’80

For $d = 2$ **and all** $\beta > \beta_c$ $\qquad \mathcal{G} = [\,\mu^-,\, \mu^+\,]$

CNPR–Theorem: Coniglio, Nappi, Peruggi, Russo '76

$$\mu \in \mathcal{G},\ \mu \neq \mu^- \ \Rightarrow\ \mu\big(\exists \text{ infinite } + \text{cluster}\big) > 0$$

Pf: Otherwise for any Δ and large Λ with prob. $\geq 1 - \varepsilon$



$\exists$ largest $\Gamma \subset \Lambda$ s.t. $\Delta \subset \Gamma$ and $\omega \equiv -1$ on $\partial\Gamma$

strong Markov property $\Rightarrow$

$$\mu = \int \gamma_\Gamma(\cdot\,|-)\, d\mu \preceq \mu^- \quad \text{on } \Delta$$

Δ arbitrary $\Rightarrow$ $\mu \preceq \mu^-$ $\Rightarrow$ $\mu = \mu^-$

Conclusion:

$$\mu \in \text{ex}\,\mathcal{G},\ \mu \neq \mu^-,\ \mu^+ \ \Rightarrow$$
$$\mu\big(\exists \text{ both infinite } + \text{ and } - \text{clusters}\big) = 1$$

Show that this is impossible in two dimensions!

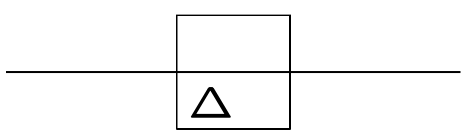
5. 2d Ising model: only two invariant phases

(Joint work with Y. Higuchi)

Butterfly lemma: $\mathcal{G}$-almost surely $\exists$ line s.t.

either ——— or ——— or both

Pf: Otherwise for some $\mu \in \mathrm{ex}\,\mathcal{G}$ and any square Δ



$\omega \le R \circ T(\omega)$ on boundary of some random region $\Gamma \supset \Delta$

($R =$ reflection, $T =$ spin flip)

$\Rightarrow \ \mu \preceq \mu \circ R \circ T$ on Δ

Similarly: $\mu \succeq \mu \circ R \circ T$ on Δ

$\Rightarrow \ \mu = \mu \circ R \circ T$ for <u>all</u> reflections R

$\Rightarrow \ \mu \ \ 2\,\mathbb{Z}^2$-invariant $\ne \mu^-, \mu^+$

CNPR & Burton–Keane $\Rightarrow$

μ-a.s. $\exists$ unique infinite $+$ and $-$clusters

Y. Zhang's argument: Positive correlations $\Rightarrow$

$$\mu\Big(\qquad\qquad \Big) > \frac{3}{4} \ \Rightarrow \ \mu\Big(\qquad\qquad \Big) > 0$$

Cor: *Only two periodic phases*

$$\boxed{\mathcal{G}_{\ominus} = [\,\mu^{-},\, \mu^{+}\,]}$$

Pf: Show: Infinite $+$ and $-$clusters cannot coexist $\mathcal{G}_{\ominus}$-a.s.

$\Big(\Rightarrow \mathcal{G}_{\ominus} \ni \mu = p_{+}\,\mu^{+} + p_{-}\,\mu^{-}$ with $p_{\pm} = \mu(\exists$ inf. $\pm$cluster$)\Big)$

Otherwise: butterfly lemma and finite energy $\Rightarrow$

$$\mu\Bigg(\qquad\qquad \Bigg) > 0$$

Poincaré recurrence $\Rightarrow$

$$\mu\Bigg(\qquad\qquad \Bigg) > 0$$

to Burton–Keane

Cor: $\mu^{+}(\exists\ +\text{sea}) = 1$

Pf: No coexistence of infinite $+*$ and $-*$ clusters $\square$

6. 2d Ising model: no non-invariant phase

Line touching lemma (Russo):

$\mathcal{G}$-almost surely, each infinite $+$ (or $+*$) cluster in a half-plane touches the boundary line infinitely often

Pf: Sufficient (by finite energy):

A.s. each infinite $+$cluster touches the boundary at least once.

If not: $\exists$ separating $-*$path π^{-*}

$$\mu\Bigg(\qquad\qquad\Bigg) > 0$$

This probability increases if

- π^{-*} is shifted down to the horizontal axis
- $+$ boundary conditions are imposed on the upper half-plane
- the $-$axis is shifted downwards ad infinitum

The limiting state is $\in \mathcal{G}_\ominus$ and a.s. admits an infinite $+$cluster, hence no infinite $-$cluster

But:

construction & reflection symmetry $\Rightarrow$ $\exists$ infinite $-$cluster

Cor: Uniqueness of infinite clusters in half-planes

$\mathcal{G}$-almost surely, each half-plane contains

at most one infinite $+$ (or $+*$) cluster

Pf: Otherwise

________________ to line touching

Prop: Percolation in half-planes

$\mathcal{G}$-almost surely in <u>each</u> half-plane

either ________________ or ________________

or ________________________

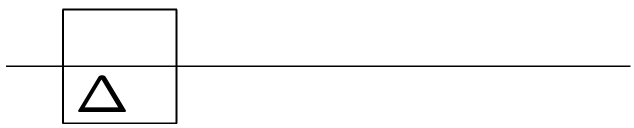
Pf: • Infinite butterflies can be shifted (random Borel-Cantelli)

• Infinite butterflies of both orientations must occur (refinement of butterfly lemma and Poincaré recurrence, using half-plane uniqueness in place of Burton–Keane) □

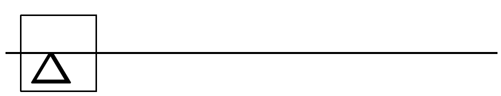
$\theta := \mu^{+}(0 \xleftrightarrow{+*} \infty) > 0$ *percolation probability*

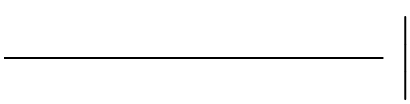
<u>*Pinning lemma*</u>: If $\mu\left(\underline{\hspace{4cm}}\right) = 1$

then for all Δ and x sufficiently far to the right

$\mu\Big($  $\Big) > \frac{\theta}{4}$

Pf:

$\mu\Big($  $\Big) > \frac{1}{2}$ for x far right

$\mu\Big($  $\Big) > \frac{1}{2}$ by reflection symmetry

$\mu\Big($ $\Big) \geq \theta$ by stochastic monotonicity

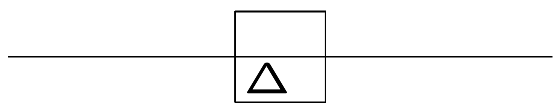
Prop: All phases are translation invariant

Each $\mu \in \mathrm{ex}\,\mathcal{G}$ is ϑ_{hor}-invariant

Pf: by Aizenman's duplication trick:

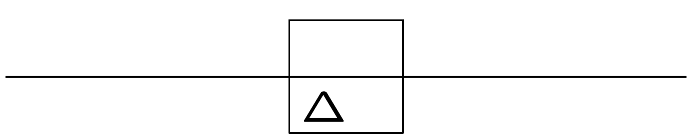
Consider $\nu = \mu \times \mu \circ \vartheta_{\mathrm{hor}}^{-1}$ *two layers of spins*

Sufficient to show:

$\forall\, \Delta \qquad \nu\Big((\omega, \omega') :$  $\Big) = 1$

For, this implies $\mu \preceq \mu \circ \vartheta_{\mathrm{hor}}^{-1}$ inside any Δ
and thus (by interchange of layers) $\mu = \mu \circ \vartheta_{\mathrm{hor}}^{-1}$

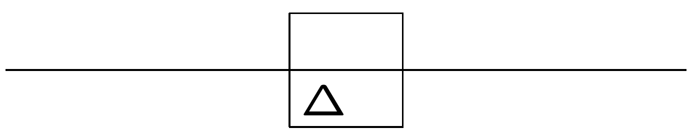
By positive correlations, reflection symmetry and triviality of the joint tail, this follows from

$\forall\, \Delta \qquad \nu\Big($  $\Big) \geq \left(\dfrac{\theta}{4}\right)^2$

Case A

μ-a.s.

pinning lemma, positive correlation $\Rightarrow$

$\mu \Bigg($  $\Bigg) \geq \left(\frac{\theta}{4}\right)^2$

$(-)$ *in first layer* $\Rightarrow$ *first* $\leq$ *second*

Case B

μ-a.s. $\Rightarrow$ $\mu \circ \vartheta_{\mathrm{hor}}^{-1}$-a.s.

Similarly: $(+)$ *in second layer* $\Rightarrow$ *first* $\leq$ *second*

Case C

μ-a.s.

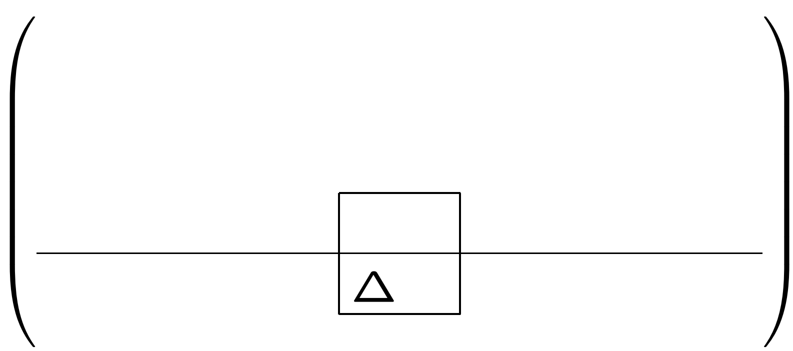
$\exists$ unique semi-infinite contour

Similarly in the second layer

The semi-infinite contours in both layers intersect each other infinitely often

Pf: Otherwise contours eventually on one side of each other.
Symmetry $\Rightarrow$ contours go eventually in parallel
But positive chance of deviations. $\square$

pinning lemma $\Rightarrow$

$\times$  $\geq$

$-*$path in first layer off Δ from x to contour intersection,
from there $+*$path off Δ in second layer to y

$\Rightarrow$ $(\leq)*$path above Δ from x to y $\square$

Epilogue: Extensions

Essential features of the lattice

- planarity
- periodicity & mirror symmetry

The statement $\mathcal{G} = [\mu^-, \mu^+]$ thus holds for the Ising model on any such lattice, e.g.

- triangular
- honeycomb
- diced
- Kagomé

Open: e.g. $(\mathbb{Z}^2)^*$ (diagonal interaction)

Essential features of the interaction

- nearest-neighbor
- FKG-attractivity
- invariance under flip–reflections

The result therefore extends to any such interaction, e.g.

- Ising model on $\mathbb{Z}^2$ in staggered external field
  $\Longleftrightarrow$ *Ising antiferromagnet for arbitrary field* $h \in \mathbb{R}$
- *hard-core lattice gas on* $\mathbb{Z}^2$
  (though finite-energy does not hold)

Open: e.g. Widom–Rowlinson lattice gas