$\Pi_3$-reflection in Kripke-Platek set theory and nonmonotone inductive definitions from the class $[\Pi_1^0, \ldots, \Pi_1^0]$.

Dieter Probst

Institut für Informatik und angewandte Mathematik, Universität Bern

München 08, Honouring Wilfried Buchholz
## Impredicative and metapredicative theories

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# Impredicative and metapredicative theories

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**$\hat{\text{ID}}_1$: $P^A$ is just *some* fixed points**

**Inductive definitions:**

$$P^A_\alpha := P^A_{<\alpha} \cup F^A(P^A_{<\alpha})$$

*restricted* ordinal induction

ordinals are just *linearly ordered*

- $K\Pi m^0$, metapredicative Mahlo
- *no* $\in$-induction
Non-monotone inductive definitions – a tool to embed admissible set theory into second order arithmetic

Remark

Ordinal analysis is simplest in subsystems of second order arithmetic
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\[
n\text{-hyper Mahlo} \quad \uparrow \quad \text{FID}[\Pi^0_1, \ldots, \Pi^0_1] \quad \text{KP} \quad \cdots \quad \text{EM} \\
\text{N+2} \quad \text{NON-MON} \quad \downarrow \quad \text{SOA} \quad ?
\]

Remark

*Ordinal analysis is simplest in subsystems of second order arithmetic*
FID(\mathcal{K}): A theory of first order inductive definitions

FID(\mathcal{K}) is formulated in a language extending L_1 by

- ordinal variables \( \alpha, \beta, \gamma, \ldots \),
- a binary relation symbol \( \alpha < \beta \),
- for each L_1(P) formula \( A(P, u) \in \mathcal{K} \), a binary relation symbol \( P^A(\alpha, x) =: P^A_\alpha(x) \).
**FID(\mathcal{K}): A theory of first order inductive definitions**

FID(\mathcal{K}) is formulated in a language extending L_1 by

- ordinal variables \( \alpha, \beta, \gamma, \ldots \),
- a binary relation symbol \( \alpha < \beta \),
- for each \( \text{L}_1(\text{P}) \) formula \( A(\text{P}, u) \in \mathcal{K} \), a binary relation symbol
  \[ P^A(\alpha, x) =: P^A_\alpha(x). \]

Further abbreviations:

- \[ P^A(s) := \exists \beta P^A_\beta(s), \]
- \[ P^A_{<\alpha}(s) := (\exists \beta < \alpha)P^A_\beta(s), \]
- \[ s \in F^A(P^A) := A(P^A, s), \]
The axioms of FID(\(\mathcal{K}\))

- The axioms of PA without induction.
- \(<\) is just a linear ordering on the ordinals with least element 0.

\[
P^A_\alpha = P^A_{\prec \alpha} \cup F^A(P^A_{\prec \alpha}).
\]

\[
s \in F^A(P^A) \rightarrow P^A(s).
\]

- Induction along N for all formulas.
- restricted ordinal induction:

\[
\forall \alpha [(\forall \beta < \alpha)(B(\beta, \delta) \rightarrow B(\alpha, \delta))] \rightarrow \forall \alpha B(\alpha, \delta),
\]

for each \(\Delta^0_0\) formula of the form \(\delta \leq \gamma \rightarrow B'(\gamma, \delta)\) with only the displayed ordinal variables free.
An important observation

Each element $s \in P^A$ enters $P^A$ for a reason:

$$P^A_\alpha(s) \land P^A_{<\alpha}(s) \rightarrow (\exists \beta < \alpha) P^A_\beta(s) \land \neg P^A_{<\beta}(s).$$
Operator forms form $[\Pi^0_1, \ldots, \Pi^0_1]$

**Definition**

$A(P, u)$ is in $[\Pi^0_1, \Pi^0_1]$, if $A_1(P, u), A_2(P, u)$ are $\Pi^0_1$, and $A(P, u)$ is as follows:

$$[F^{A_1}(P) \not\subseteq P \land A_1(P, u)] \lor [F^{A_1}(P) \subseteq P \land A_2(P, u)].$$

**Definition**

$A(P, u)$ is in $[\Pi^0_1, \ldots, \Pi^0_n]$, if $A_1(P, u), \ldots, A_n(P, u)$ are $\Pi^0_1$, and $A(P, u)$ is as follows:

$$\bigvee_{1 \leq i \leq n} \left[ \bigwedge_{j < i} (F^{A_j}(P) \subseteq P) \land F^{A_i}(P) \not\subseteq P \land A_i(P, u) \right].$$
KPU^0 and KPm^0: KPU and KPm without foundation

KPU^0 is formulated in L_1(\in, N). It formalizes a universe of sets above the natural numbers N, which are the urelements.

- For all axioms A(\bar{a}) of PA except induction, \bar{a} \in N \rightarrow A^N(\bar{a}) is an axiom of KPU^0.
- Kripke-Platek axioms: Pair, union, \Delta_0-separation, \Delta_0-collection.
- Complete induction on the natural numbers for \textit{sets}.

An admissible set is a transitive model of KPU^0. KPm^0 is an extension of KPU^0 formulated in L_1(\in, N, Ad). It formalizes the existence of

- arbitrary large admissible sets that are \textbf{linearly ordered by} \in.
- \Pi_2\text{-reflection on admissibles.}
$n$-hyper Mahlo

Let $\pi_2(e, x)$ be a universal $\Pi_2$ formula. $A^a$ is the formula obtained from $A$ by replacing each quantifier $Qx$ in $A$ by $(Qx \in a)$. Occasionally, we write $a \models A$ for $A^a$. and $a \in \text{Ad}$ for $\text{Ad}(a)$. 
**n-hyper Mahlo**

Let $\pi_2(e, x)$ be a universal $\Pi_2$ formula. $A^a$ is the formula obtained from $A$ by replacing each quantifier $Qx$ in $A$ by $(Qx \in a)$. Occasionally, we write $a \models A$ for $A^a$. and $a \in Ad$ for $Ad(a)$.

### Definition

- $Ad_0(a) := Ad(a) \land \forall x \in a (\exists b \in a)(x \in b \land Ad(b))$, and $Ad_{n+1}(a)$ is
  
  $$a \in Ad(\forall e \in N)[a \models \pi_2(x, e) \rightarrow (\exists b \in Ad_n \cap a)(x \in b \land b \models \pi_2(x, e))].$$

  If $Ad_0(a)$, then we say that $a$ is 1-inaccessible,
  if $Ad_{n+1}(a)$, then we say that $a$ is $n$-hyper Mahlo.

- 0-hyper Mahlo is $KPM^0$ and $n+1$-hyper Mahlo is $KPU^0$ plus
  $\Pi_2$-reflection on admissibles that are $n$-hyper Mahlo:

  $$\forall e \in N)[\pi_2(x, e) \rightarrow (\exists b \in Ad_n)(x \in b \land b \models \pi_2(x, e))].$$
Parameter-free $\Delta$-induction along $(a \cap \text{Ad}, \in)$

**Lemma**

Assume that $A(u, v)$ is a $\Delta$ formula with only the displayed variables free. If

$$\emptyset \neq C := \{ x \in \text{Ad} : A(x, d) \} \text{ and } (\forall x \in C)(d \in x),$$

then $C$ has an $\in$-least element.
Lemma

Assume that $A(u, v)$ is a $\Delta$ formula with only the displayed variables free. If

$$\emptyset \neq C := \{x \in \text{Ad} : A(x, d)\} \quad \text{and} \quad (\forall x \in C)(d \in x),$$

then $C$ has an $\in$-least element.

Assume that $C$ has no $\in$-least element.

- For each $a \in C$, there is an $b \in C$ with $b \in a$. Further,
  $$\bigcap C = \bigcap (C \cap b) \in a.$$  
- Hence $c := \bigcap C$ is a set, and $c \in c$.
- $c$ is an intersection of admissible and thus satisfies $\Delta$-separation.

Therefore, $r := \{x \in c : x \notin x\} \in c$, and so $r \in r \iff r \notin r$!
A first application – hierarchies along \((a \cap \text{Ad}, \in)\)

For each \(L_1(P)\) formula \(A(P, u)\), \(\text{Hier}^A(f, a)\) is the conjunction of the following two \(\Delta_0\) formulas:

- \(\text{Dom}(f) = a \cap \text{Ad},\)

- \((\forall x \in \text{Dom}(f))[f(x) = f_{<x} \cup \{n \in \mathbb{N} : A^\mathbb{N}(f_{<x}, n)\}],\)

where \(f_{<x} := \bigcup_{y \in x} f(y)\). Otherwise, \(\{a \in \text{Ad} : \neg(\exists f \in a^+)\text{Hier}^A(f, a)\}\) had a \(\in\)-least element!
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where \(f_{<x} := \bigcup_{y \in x} f(y)\).

**Lemma (Provable in KPm\(^0\))**

*For each \(L_1(P)\) formula \(A(P, u)\),

\[(\forall a \in \text{Ad})(\exists! f \in a^+)\text{Hier}^A(f, a),\]

where \(a^+\) denotes the \(\in\)-least element of the non-empty class \(\{x \in \text{Ad} : a \in x\}\).*

Otherwise, \(\{a \in \text{Ad} : \neg(\exists f \in a^+)\text{Hier}^A(f, a)\}\) had a \(\in\)-least element!
A first application – hierarchies along \((a \cap \text{Ad}, \in)\)

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where \(f_{<x} := \bigcup_{y \in x} f(y)\).

**Lemma (Provable in KPm⁰)**

For each \(L_1(P)\) formula \(A(P, u)\),

\[(\forall a \in \text{Ad})(\exists! f \in a^+)\text{Hier}^A(f, a),\]

where \(a^+\) denotes the \(\in\)-least element of the non-empty class \(\{x \in \text{Ad} : a \in x\}\).

Otherwise, \(\{a \in \text{Ad} : \neg(\exists f \in a^+)\text{Hier}^A(f, a)\}\) had a \(\in\)-least element!
Embedding FID$[\Pi^0_1, \Pi^0_1]$ into $n$-hyper Mahlo

If $A(P, u)$ is an $L_1(P)$ formula, then for each admissible $a$, $P^A_a := f(a)$ and $P^A_{<a} := f_{<a}$, where $f \in a^+$ is the unique function satisfying Hier$^A(f, a^+)$. 
Embedding $\text{FID}[^{\Pi_0^0, \Pi_1^0}_n]$ into $n$-hyper Mahlo

If $A(P, u)$ is an $L_1(P)$ formula, then for each admissible $a$, $P^A_a := f(a)$ and $P^A_{<a} := f_{<a}$, where $f \in a^+$ is the unique function satisfying $\text{Hier}^A(f, a^+)$. 

**Lemma (Provable in KPm$^0$) (auxiliary lemma)**

Let $C(U^+, V^-, u)$ be a $\Pi_1^0$ formula of $L_1$ and $a \in \text{Ad}$. Then,

$$(\forall b \in a) C^N(P^A_{<a}, P^A_b, n) \rightarrow C^N(P^A_{<a}, P^A_{<a}, n).$$
Let $x \in a := x \in a \cap \text{Ad}$.

**Lemma (Provable in KPm$^0$)**  
(\(\Pi_2\)-reflection on stages))

Let $A(u, v)$ be $\Delta$ such that $x, y, z \in \text{Ad} \land y \in z \land A(x, y) \rightarrow A(x, z)$ and $a$ 1-inaccessible. Then,

$$
(\forall x \in a)(\exists y \in a)A(x, y) \rightarrow (\exists b \in a)(\forall x \in b)(\exists y \in b)A(x, y).
$$
Let $x \in a := x \in a \cap \text{Ad}$.

**Lemma (Provable in KPM$^0$) (Π$_2$-reflection on stages))**

Let $A(u, v)$ be $\Delta$ such that $x, y, z \in \text{Ad} \land y \in z \land A(x, y) \rightarrow A(x, z)$ and a $1$-inaccessible. Then,

$$(\forall x \in a)(\exists y \in a)A(x, y) \rightarrow (\exists b \in a)(\forall x \in b)(\exists y \in b)A(x, y).$$

- Assuming the premise, let $f(0) := \emptyset^+$ and, if $f(n) = c \in \text{Ad}$, then $f(n+1)$ is the $\in$-least admissible of the class

$$\{z \in \text{Ad} : c \in z \land (\forall x \in c)(\exists y \in z)A(x, y)\}.$$

- $f$ is in $a$. Further, we can view $f(n)$ as a $\Sigma$-function symbol. Now let $b$ be the $\in$-least element of the class

$$\{z \in \text{Ad} : (\forall n \in \mathbb{N})(f(n) \in z)\}.$$
Lemma (Provable in $\text{KPm}^0$) (base case)

Let $A(P, u)$, $B(P, u)$ be $L_1(P)$ formulas with $B \Pi^0_1$. Suppose that $a$ is 1-inaccessible. Then

$$B^N(P^A_{<a}, n) \rightarrow (\exists b \in a) B^N(P^A_{<b}, n).$$
Lemma (Provable in KPm$^0$) (base case))

Let $A(P, u), B(P, u)$ be $L_1(P)$ formulas with $B \Pi^0_1$. Suppose that $a$ is $1$-inaccessible. Then

$$B^N(P^A_{<a}, n) \rightarrow (\exists b \in a)B^N(P^A_{<b}, n).$$

- Let $C(U^+, V^-, u)$ s. t. $B(X, x) \leftrightarrow C(X, X, x)$. Assume $A^N(P^A_{<a}, n)$.
- $(\forall b \in a)C^N(P^A_{<a}, P^A_b, n)$ \hspace{1cm} (V$^-$)
- $(\forall b \in a)(\exists c \in a)C^N(P^A_c, P^A_b, n)$ ($\Sigma$ reflection in $a$).
- $(\forall b \in d)(\exists c \in d)C^N(P^A_c, P^A_b, n)$ for some $d \in a$. \hspace{1cm} ($\Pi_2$-refl. Lemma)
- $(\forall b \in d)C^N(P^A_{<d}, P^A_b, n)$ \hspace{1cm} (U$^+$)
- By the aux. Lemma: $C^N(P^A_{<d}, P^A_{<d}, n)$, i.e. $B^N(P^A_{<d}, n)$. 

D. Probst (IAM, Uni Bern)
Lemma (Provable in KPm$^0$)

Let $A(P,u)$ be an operator form from $[\Pi^0_1, \ldots, \Pi^0_1]$ with components $A_0, \ldots, A_n$. For all $k \leq n$, if $a \in \text{Ad}_k$, then for all $i \leq k$,

$$A_i^N(P^A_{<a}, n) \rightarrow n \in P^A_{<a}.$$
Lemma (Provable in KPm\(^0\))

Let \(A(P, u)\) be an operator form from \([\Pi_0^1, \ldots, \Pi_0^1]\) with components \(A_0, \ldots, A_n\). For all \(k \leq n\), if \(a \in Ad_k\), then for all \(i \leq k\),

\[
A_i^N(P^A_{<a}, n) \rightarrow n \in P^A_{<a}.
\]

- Let \(C_i(U^+, V^-, u)\) s. t. \(A_i(X, x) \leftrightarrow C_i(X, X, x)\). Assume \(A_i(P^A_{<a}, n)\).
- \(k = 0\): if \(A_0(P^A_{<a}, n)\), then \(A_0(P^A_{<b}, n)\) for some \(b \in a\), and \(n \in P^A_b\) whether \(A_0\) is active at stage \(b\) or not.
- \(k \rightarrow k+1\): As before, \((\forall b \in a)(\exists c \in a)C_i^N(P^A_c, P^A_b, n)\).

Since \(a \in Ad_{k+1}\), there is an \(a' \in Ad_k\) with

\[
(\forall b \in a')(\exists c \in a')C_i^N(P^A_c, P^A_b, n).
\]

- \(i \leq k\): \(n \in P^A_{<a'}\) by the I.H.
- \(i = k+1\): \(A_{k+1}\) is active at stage \(a'\) thus \(n \in P^A_{a'}\).
Remark

- $\text{KP}^0 + \exists a \text{Ad}_{n+1}(a)$ is $\text{KP}^0$ above a model of $n$-hyper Mahlo.

\[ |\text{KP}^0 + \exists a \text{Ad}_{n+1}(a)| = |n\text{-hyper Mahlo} + \text{fml ind. along } \mathbb{N}|. \]
Remark

- $\text{KPU}^0 + \exists a \text{Ad}_{n+1}(a)$ is $\text{KPU}^0$ above a model of $n$-hyper Mahlo.

  $|\text{KPU}^0 + \exists a \text{Ad}_{n+1}(a)| = |n$-hyper Mahlo + fml ind. along $\mathbb{N}|$.

Theorem

$|\text{FID}[\Pi^0_1, \ldots, \Pi^0_1]| \leq |n$-hyper Mahlo + fml ind$, and$

|\bigcup_n \text{FID}[\Pi^0_1, \ldots, \Pi^0_1]| \leq |\text{KPU}^0 + \Pi^1_3$-Refl$_\text{Ad}|$. 
Non-monotone inductive definitions – a tool to embed set theory into SOA?

Remark

*Ordinal analysis is simplest in subsystems of second order arithmetic*
A family of theories $T_{\vec{\alpha}}$ build by two operations

Let $C(U)$ be an $L_2$ formula and $\pi_2^1(U, u)$ a universal $\Pi^1_2$ formula. Then

- **Limit:** $l(C) := \forall X \exists Y[X \in Y \land Y \models C(X)]$.
- **$\Pi^1_2$-reflection:** $p(C)$ is the universal closure of
  \[
  \pi_2^1(X, e) \rightarrow \exists Y[X \in Y \land Y \models C(X) \land Y \models \pi_2^1(X, e)].
  \]

Next, for $\alpha, \beta, \gamma, \ldots$ below $\Phi_0$, we assign the theories as follows:

### The theories of the form $T_{\alpha, \beta, \gamma}$ and $T_{\alpha, \Phi_0, 0}$

- $T^0(\emptyset) := \exists Y[Y \models \exists X(X = \{x : \pi_1^0(\emptyset, x, e)\})]$,
- $T^{\alpha, \beta, \gamma} := l^\gamma \circ (l \circ p^1)^\beta \circ (l \circ p^2)^\alpha (T^0)$,
- $T^{\alpha, \Phi_0, 0} := (p^2) \circ (l \circ p^2)^\alpha (T^0)$. 
Future work

**Theorem**

Let \( 0 < \Phi_0 \) be the least ordinal closed under all \( n \)-ary Veblen functions \( \varphi^n \). Then we have for all \( \alpha_1, \ldots, \alpha_{k-1} < \Phi_0 \),

- \( |T^{\alpha_{k-1}, \ldots, \alpha_1}| = \varphi^k \alpha_{k-1} \ldots \alpha_1 0 \).
- \( |T^{\alpha_{k-1}, \ldots, \alpha_i, \Phi_0, \vec{0}}| = \varphi^k \alpha_{k-1} \ldots \alpha_i \omega \vec{0} 0 \).

**Example**

- The theory \( T^{1,0} \) is the limit of \( p(T^0) \simeq \Sigma_1^1 \)-DC\(_0\). \( |\text{ATR}_0| = \varphi^3 100 = \Gamma_0 \).
- \( T^{1, \Phi_0} \) formalizes \( \Pi_2^1 \)-reflection on models of \( \text{ATR}_0 \). \( |\text{ATR}_0 + (\Sigma_1^1 \text{-DC})| = \varphi^3 1\omega 0 \).
- \( T^{\Phi_0, 0} \) is \( p^2(T^0) = p(T^{\Phi_0}) \). \( |\Pi_2^1 \text{-Refl}(_{\Sigma_1^1 \text{-DC}})| = \varphi^3 \omega 00 \).
**Definition**

Let $C$ be some $L_2$ sentence. Then

$$\Pi^1_n\text{-Refl}_C(W, e) := \pi^1_n(W, e) \rightarrow \exists M[W \in M \land M \models C \land M \models \pi^1_n(W, e)],$$

is an instance of $\Pi^1_n$ reflection on models of $C$.

Further, $\Pi^1_n\text{-Refl}_C := \forall X, x \Pi^1_n\text{-Refl}_C(X, x)$, which is a $\Pi^1_{n+1}$ sentence.

**Remark (The contraposition of $\Pi^1_n\text{-Refl}_C(W, e)$)**

Suppose that $A(U)$ is $\Sigma^1_n$. Then

$$\forall M[W \in M \land M \models C \rightarrow A^M(W)] \rightarrow A(W), \text{ i.e.}$$

$$W \notin M, M \not\models C, A^M(W), A(W).$$
Iterated $\Pi^1_n$-reflection exhausts $\Pi^1_{n+1}$-reflection

**Definition (Iterated $\Pi^1_n$-reflection)**

- $S^0_n := (\text{ACA}) := \forall X, e \exists Y [Y = \{x : \pi^0_1(X, e, x)\}]$,
- $S^k+1_n := \Pi^1_n$-$\text{Refl}_{S^k_n}$.

**Lemma**

*If $\Gamma$ is a finite set of $\Sigma^1_n$ formulas, then*

$$\text{ACA}_0 + \Pi^1_{n+1}$-$\text{Refl}_{(\text{ACA})} \mid^k \Gamma \implies S^k_n \vdash \Gamma.$$
Future work

$T \vdash \Gamma$ is obtained by a cut with $[-] \Pi^1_{n+1} \text{Refl}_{(ACA)}$.

Assume that $\pi^1_{n+1}(U, s) = \forall XB(X, U)$.

$\land$- and $\forall$-inversion and the I.H. yield

\[
S^k_n \vdash \Gamma, \exists Y B(Y, U), \\
S^k_n \vdash \Gamma, U \notin M, M \not\models (ACA), M \not\models \forall XB(X, U).
\]

In $S^k_{n+1}$ we have models of $S^k_n$ above arbitrary sets:

\[
S^k_{n+1} \vdash \vec{W}, U \notin M, M \not\models S^k_n, \Gamma^M, M \models \forall XB(X, U), \\
S^k_{n+1} \vdash \Gamma, U \notin M, M \not\models S^k_n, M \not\models \forall XB(X, U).
\]

A cut yields $S^k_{n+1} \vdash \vec{W}, U \notin M, \Gamma, \Gamma^M, M \models S^k_n$.

By contraposition of $\Pi^1_{n} \text{Refl}_{S^k_n}$ we get $S^k_{n+1} \vdash \Gamma$. 

Read a formula of $\text{ID}_1$ as an $L_2$ formula by reading $s \in P^A$ as an abbreviation for $\forall X[F^A(X) \subseteq X \rightarrow s \in X]$.

**Lemma**

Suppose that $\text{ID}_1$ proves $\Gamma$ and that (the translation of ) all formulas that occur in the proof-tree are at most $\Sigma^1_n$.

$$\text{ID}_1 \vdash_k \Gamma \iff \text{ACA} + S^k_n \vdash \Gamma.$$
Read a formula of $\text{ID}_1$ as an $L_2$ formula by reading $s \in P^A$ as an abbreviation for $\forall X[F^A(X) \subseteq X \rightarrow s \in X]$.

**Lemma**

Suppose that $\text{ID}_1$ proves $\Gamma$ and that (the translation of ) all formulas that occur in the proof-tree are at most $\Sigma^1_n$.

$$\text{ID}_1 \mid_{k}^{*} \Gamma \implies \text{ACA} + S^k_n \vdash \Gamma.$$

- Have: $\text{ID}_1 \mid_{k+1}^{*} \Gamma, P^A \subseteq F^B(P^A)$.
- By I.H. $S^k_n \mid_{k}^{*} \Gamma, F^A(F^B(P^A)) \subseteq F^B(P^A)$.
- $S^k_{n+1} \vdash M \nvdash S^k_n, \Gamma^M, F^A(F^B(P^A \upharpoonright M)) \subseteq F^B(P^A \upharpoonright M)$.
- $P^A \upharpoonright M$ is a set in $S^k_{n+1}: S^k_{m+1} \vdash M \nvdash S^k_n, \Gamma^M, s \notin P^A, s \in F^B(P^A \upharpoonright M)$.
- Now $\Pi^1_n$-Refl$_S^n$ yields $S^k_{n+1} \vdash \Gamma, s \notin P^A, s \in F^B(P^A)$. 
Future work

\[ \bigcup_{n \in \mathbb{N}} \text{FID}(\Pi^0_n) \]

\[ \bigcup_{n \in \mathbb{N}} \Pi^1_n\text{-Refl}_{\text{ACA}_0} \]

\[ \Psi \varepsilon \Omega + 1 \]