The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

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What is the modal $\mu$-calculus?
What is the modal $\mu$-calculus?

The modal $\mu$-calculus...  
... is an extension of modal logic allowing least and greatest fixpoint constructors for any (syntactically) monotone formula containing "all" extensions of modal logic with fixpoint constructors.
What is the modal $\mu$-calculus?

The modal $\mu$-calculus...

... is an extension of modal logic allowing least and greatest fixpoint constructors for any (syntactically) monotone formula, containing ”all” extensions of modal logic with fixpoint constructors.

- **PDL:** $\langle \alpha^* \rangle \psi = \mu x. \psi \lor \langle \alpha \rangle x$
- **CTL:** $\text{EG} \varphi = \nu x. \varphi \land \Diamond x$ and $\text{E}(\varphi U \psi) = \mu x. \psi \lor (\varphi \land \Diamond x)$
Some expressible properties

Eventually ”p”:
Some expressible properties

Eventually "p":

\[ \mu x. p \lor \lozenge x \]
Some expressible properties

Eventually "p":

\[ \mu x. p \lor \Diamond x \]

Allways "p":


Some expressible properties

Eventually "p":

$$\mu x.p \lor \Diamond x$$

Allways "p":

$$\nu x.p \land \Box x$$
Some expressible properties

Eventually "p":
\[ \mu x. p \lor \Diamond x \]

Allways "p":
\[ \nu x. p \land \Box x \]

Allways eventually "p":
\[ \nu x. (\mu y. p \lor \Diamond y) \land \Box x \lor \Diamond y \]
Some expressible properties

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Some expressible properties

Eventually ”p”:
$$\mu x.p \lor \Diamond x$$

Allways ”p”:
$$\nu x.p \land \Box x$$

Allways eventually ”p”:
$$\nu x.(\mu y.p \lor \Diamond y) \land \Box x$$

There is a branch such that infinitely often ”p”:
Some expressible properties

Eventually "p":
$$\mu x. p \lor \Box x$$

Always "p":
$$\nu x. p \land \Box x$$

Always eventually "p":
$$\nu x. (\mu y. p \lor \Box y) \land \Box x$$

There is a branch such that infinitely often "p":
$$\nu x. \mu y. (p \land \Box x) \lor \Box y$$
Fixpoint alternation depth "ad"
Fixpoint alternation depth "ad"

Eventually "p" and allways "p":

$$\text{ad}(\mu x. p \lor \Diamond x) = \text{ad}(\nu x. p \land \Box x) = 1$$
Fixpoint alternation depth "ad"

Eventually "p" and allways "p":

\[ \text{ad}(\mu x. p \lor \Box x) = \text{ad}(\nu x. p \land \Box x) = 1 \]

There is a branch such that infinitely often "p":

\[ \text{ad}(\nu x. \mu y. (p \land \Box x) \lor \Diamond y) = 2 \]
Fixpoint alternation depth "ad"

Eventually "p" and allways "p":

\[ \text{ad}(\mu x. p \lor \Box x) = \text{ad}(\nu x. p \land \Box x) = 1 \]

There is a branch such that infinitely often "p":

\[ \text{ad}(\nu x. \mu y. (p \land \Box x) \lor \Box y) = 2 \]

\[ \Rightarrow \text{the internal fixpoint formula } \mu y. (p \land \Box x) \lor \Box y \text{ uses the external fixpoint variable } x \text{ as parameter.} \]
Fixpoint alternation depth "ad"

Eventually "p" and allways "p":

\[
ad(\mu x. p \lor \lozenge x) = ad(\nu x. p \land \Box x) = 1
\]

There is a branch such that infinitely often "p":

\[
ad(\nu x. \mu y. (p \land \lozenge x) \lor \lozenge y) = 2
\]

⇒ the internal fixpoint formula \( \mu y. (p \land \lozenge x) \lor \lozenge y \) uses the external fixpoint variable \( x \) as parameter.

Allways eventually "p":

\[
ad(\nu x. (\mu y. p \lor \lozenge y) \land \Box x) = 1
\]
A formula with ad = 3:

\[ \varphi \equiv \mu x. \nu y. \mu z. ((d_1 \land \Diamond x) \lor (d_2 \land \Diamond y) \lor (d_3 \land \Diamond z) \lor \ldots 
\]

\[ \ldots \lor (c_1 \land \Box x) \lor (c_2 \land \Box y) \land (c_3 \land \Box z)) \]

\[ \Rightarrow \text{the subformula } \varphi_z \text{ uses the fixpoint variable } y \text{ as parameter and} \]

\[ \text{the subformula } \varphi_y \text{ uses the most external fixpoint variable } x \text{ as parameter.} \]
A formula with $\text{ad} = 3$:

$$\varphi \equiv \mu x. \nu y. \mu z. ((d_1 \wedge \Diamond x) \lor (d_2 \wedge \Diamond y) \lor (d_3 \wedge \Diamond z) \lor \ldots$$

$$\ldots \lor (c_1 \wedge \Box x) \lor (c_2 \wedge \Box y) \lor (c_3 \wedge \Box z))$$

$\Rightarrow$ the subformula $\varphi_z$ uses the fixpoint variable $y$ as parameter and the subformula $\varphi_y$ uses the most external fixpoint variable $x$ as parameter.

**Syntactical modal $\mu$-calculus hierarchy**

The alternation depth implies a "strict" syntactical hierarchy on the class of all $\mu$-formulae.
The modal $\mu$-calculus hierarchy on Restricted Classes of Transition Systems

Introduction

The modal $\mu$-calculus hierarchy

Bradfield (1996): Stictness of semantical modal $\mu$-calculus hierarchy

The semantical modal $\mu$-calculus hierarchy is strict on the class of all transition systems.

$\Rightarrow$ For each $n$ there is a formula $\varphi$ with $\text{ad}(\varphi) = n$ such that for all formulae $\psi$ with $\text{ad}(\psi) < n$ we do not have

for all transition systems $\mathcal{T}$: $(\mathcal{T} \models \varphi \iff \mathcal{T} \models \psi)$. 

We answer the three following questions:

Strictness of the semantical modal $\mu$-calculus hierarchy on the class of all...

1. ... reflexive transition systems?
2. ... transitive and symmetric transition systems?
3. ... transitive transition systems?
Overview

Introduction

The modal $\mu$-calculus

Games for the modal $\mu$-calculus

The Hierarchy on Reflexive Transition Systems

The Hierarchy on transitive and symmetric Transition Systems

The Hierarchy on transitive Transition Systems
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The modal $\mu$-calculus

$L_\mu$-formulae

$$\varphi ::= p \mid \neg p \mid T \mid \bot \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid \Box \varphi \mid \Diamond \varphi \ldots$$

$$\ldots \mid \mu x. \varphi \mid \nu x. \varphi$$

where $p, x \in P$ and $x$ occurs only positively in $\eta x. \varphi$ ($\eta = \nu, \mu$).
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The modal $\mu$-calculus

$L_\mu$-formulae

$$\varphi ::= p \mid \sim p \mid \top \mid \bot \mid (\varphi \land \varphi) \mid (\varphi \lor \varphi) \mid \Diamond \varphi \mid \Box \varphi \ldots$$

$$\ldots \mid \mu x. \varphi \mid \nu x. \varphi$$

where $p, x \in P$ and $x$ occurs only positively in $\eta x. \varphi$ ($\eta = \nu, \mu$).

$\neg \varphi$ is defined by using de Morgan dualities for boolean connectives, the usual modal dualities for $\Diamond$ and $\Box$, and

$$\neg \mu x. \varphi(x) \equiv \nu x. \neg \varphi(x)[x/\neg x] \quad \text{and} \quad \neg \nu x. \varphi(x) \equiv \mu x. \neg \varphi(x)[x/\neg x].$$
\[ x \in \text{bound}(\varphi) \text{ then } \varphi_x \text{ is subformula of } \varphi \text{ of the form } \eta x.\alpha. \]

\[ \varphi \text{ well-named} \text{ if no two distincts occurrences of fixed point operators in } \varphi \text{ bind the same variable, no variable has both free and bound occurrences in } \varphi \text{ and if for any subformula } \eta x.\alpha \text{ of } \varphi \text{ we have that } x \text{ appears once in } \alpha. \]
Syntactical modal $\mu$-calculus hierarchy

Let $\Phi \subseteq \mathcal{L}_\mu$. $\nu(\Phi)$ is the smallest class of formulae such that:

- $\Phi, \neg\Phi \subseteq \nu(\Phi)$;
- If $\psi(x) \in \nu(\Phi)$ and $x$ occurs only positively, then $\nu x.\psi \in \nu(\Phi)$;
- If $\psi, \varphi \in \nu(\Phi)$, then $\psi \land \varphi, \psi \lor \varphi, \Diamond \psi, \Box \psi \in \nu(\Phi)$;
- If $\psi, \varphi \in \nu(\Phi)$ and $x$ is bound in $\psi$, then $\varphi[x/\psi] \in \nu(\Phi)$
Syntactical modal $\mu$-calculus hierarchy

Let $\Phi \subseteq \mathcal{L}_\mu$. $\nu(\Phi)$ is the smallest class of formulae such that:

1. $\Phi, \neg \Phi \subseteq \nu(\Phi)$;
2. If $\psi(x) \in \nu(\Phi)$ and $x$ occurs only positively, then $\nu x. \psi \in \nu(\Phi)$;
3. If $\psi, \varphi \in \nu(\Phi)$, then $\psi \land \varphi, \psi \lor \varphi, \Diamond \psi, \Box \psi \in \nu(\Phi)$;
4. If $\psi, \varphi \in \nu(\Phi)$ and $x$ is bound in $\psi$, then $\varphi[x/\psi] \in \nu(\Phi)$

similarly for $\mu(\Phi)$
For all $n \in \mathbb{N}$, we define the class of $\mu$-formulae $\Sigma^\mu_n$ and $\Pi^\mu_n$ inductively as follows:

- $\Sigma_0^\mu := \Pi_0^\mu := \mathcal{L}_M$;
- $\Sigma_{n+1}^\mu = \mu(\Pi_n^\mu)$;
- $\Pi_{n+1}^\mu = \nu(\Sigma_n^\mu)$.

Alternation depth:

$$\Delta_n^\mu := \Sigma_n^\mu \cap \Pi_n^\mu$$

$$\text{ad}(\varphi) := \inf\{k : \varphi \in \Delta_{k+1}^\mu\}.$$
Transition Systems
Transition Systems

A transition system $\mathcal{T}$ is a triple $(S, \rightarrow^\mathcal{T}, \lambda)$ consisting of
- a set $S$ of states,
- a binary relation $\rightarrow^\mathcal{T} \subseteq S \times S$ called transition relation,
- the valuation $\lambda : P \rightarrow \wp(S)$ assigning to each propositional variable $p$ a subset $\lambda(p)$ of $S$. 
Transition Systems

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A pointed transition system $(\mathcal{T}, s_0)$ consists of a transition system $\mathcal{T}$ and a distinguished state $s_0$. 
Denotation of a formula

\[ \| \varphi \|_T \] is defined as usual by induction on the complexity of \( \varphi \in \mathcal{L}_\mu \). Simultaneously for all transition systems \( T \) we set:

- \( \| \nu x. \alpha \|_T = \bigcup \{ S' \subseteq S \mid S' \subseteq \| \alpha(x) \|_{T[x \mapsto S']} \} \)
- \( \| \mu x. \alpha \|_T = \bigcap \{ S' \subseteq S \mid \| \alpha(x) \|_{T[x \mapsto S']} \subseteq S' \} \)
Denotation of a formula

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- \( \| \nu x. \alpha \|_T = \bigcup \{ S' \subseteq S \mid S' \subseteq \| \alpha(x) \|_T[x \mapsto S'] \} \)
- \( \| \mu x. \alpha \|_T = \bigcap \{ S' \subseteq S \mid \| \alpha(x) \|_T[x \mapsto S'] \subseteq S' \} \)

\[ \| \nu x. \varphi(x) \|_T = GFP(\| \varphi(x) \|_T) \] and \( \| \mu x. \varphi(x) \|_T = LFP(\| \varphi(x) \|_T) \)
Some equivalences

▶ If $x$ is not in the scope of a modality in $\varphi(x)$ then for all $T$

$$\|\nu x. \varphi(x)\|_T = \|\varphi(\top)\|_T \text{ and } \|\mu x. \varphi(x)\|_T = \|\varphi(\bot)\|_T$$

▶ For all $\varphi(x, y)$ and all $T$

$$\|\nu x. \nu y. \varphi(x, y)\|_T = \|\nu x. \varphi(x, x)\|_T$$

$$\|\mu x. \mu y. \varphi(x, y)\|_T = \|\mu x. \varphi(x, x)\|_T.$$

▶ Every formula $\varphi$ is equivalent to well-named formula $\text{nf}(\varphi)$. 
Classes of Transition Systems

- $\|\varphi\| = \{(T, s) ; s \in \|\varphi\|_T\}$
- $\|\varphi\|^r = \{(T, s) ; s \in \|\varphi\|_T \text{ and } T \text{ reflexive}\}$
- Similarly form $\|\varphi\|^t, \|\varphi\|^{st}, \|\varphi\|^{rst}$. 
For all $n \in \mathbb{N}$, we define the following classes pointed transition systems

- $\Sigma_{\mu}^n, T = \{ \parallel \varphi \parallel ; \varphi \in \Sigma_{\mu}^n \}$
- $\Pi_{\mu}^n, T = \{ \parallel \varphi \parallel ; \varphi \in \Pi_{\mu}^n \}$
- $\Delta_{\mu}^n, T = \{ \parallel \varphi \parallel ; \varphi \in \Delta_{\mu}^n \}$

Similarly for $T^r$, $T^t$, $T^{st}$ and $T^{rst}$. 
How do we decide if $s \in \| \varphi \|_T$?
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Games for the modal $\mu$-calculus
Evaluation game for classical propositional logic

\[ \mathcal{E}((q \lor r) \land p, (T, s_1)) : \]

Diagram:

- \( s_2^\{p\} \) to \( s_3^\{p\} \) to \( s_1^\{p, r\} \) to \( s_4^\{r\} \)
- \( s_2^\{p\} \) to \( s_3^\{p\} \) to \( s_4^\{r\} \)
- \( \langle (q \lor r) \land p, s_1^{} \rangle \)
- \( \langle p, s_1^{} \rangle \)
- \( \langle q \lor r, s_1^{} \rangle \)
- \( \langle r, s_1^{} \rangle \)
<table>
<thead>
<tr>
<th>position</th>
<th>player</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(&lt;p_i, s&gt;)</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(&lt;\psi \lor \phi, s&gt;)</td>
<td>V chooses between (&lt;\psi, s&gt;) and (&lt;\phi, s&gt;)</td>
<td>V choice</td>
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<tr>
<td>(&lt;\psi \land \phi, s&gt;)</td>
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<td>F choice</td>
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Games for the modal $\mu$-calculus

Evaluation game for modal logic

$E(\Diamond \Box \bot, (\mathcal{T}, s_1))$:
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Games for the modal $\mu$-calculus

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<td>$V$ chooses between $\langle \psi, s \rangle$ and $\langle \phi, s \rangle$</td>
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<td>$\langle \psi \land \phi, s \rangle$</td>
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<td>$F$ choice</td>
</tr>
<tr>
<td>$\langle \Diamond \psi, s \rangle$</td>
<td>$V$ chooses a point $s'$ s.t. $s \rightarrow s'$</td>
<td>$\langle \psi, s' \rangle$</td>
</tr>
<tr>
<td>$\langle \Box \psi, s \rangle$</td>
<td>$F$ chooses a point $s'$ s.t. $s \rightarrow s'$</td>
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### Evaluation game for the modal $\mu$-calculus

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<td>$F$ chooses between $\langle \psi, s \rangle$ and $\langle \phi, s \rangle$</td>
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</tr>
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<td>$\langle \lozenge \psi, s \rangle$</td>
<td>$V$ chooses a point $s'$ s.t. $s \rightarrow s'$</td>
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</tr>
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<td>$\langle \mu x. \psi, s \rangle$</td>
<td>-</td>
<td>$\langle \psi, s \rangle$</td>
</tr>
<tr>
<td>$\langle \nu x. \psi, s \rangle$</td>
<td>-</td>
<td>$\langle \psi, s \rangle$</td>
</tr>
<tr>
<td>$\langle x, s \rangle$</td>
<td>-</td>
<td>$\langle \psi_x, s \rangle$</td>
</tr>
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</table>
"There is an infinite branch" $E(\nu x.\diamond x, (T, s_1))$: 

```
s_2\{p\} \rightarrow s_1\{p,r\} \\
  \downarrow \\
  \downarrow \\
  \downarrow \\
\langle \nu x.\diamond x, s_1 \rangle \\
\downarrow \\
\langle \diamond x, s_1 \rangle \\
\downarrow \\
\langle x, s_3 \rangle \\
\downarrow \\
\langle \nu x.\diamond x, s_3 \rangle \\
\downarrow \\
\langle x, s_4 \rangle 
```
"There is branch with infinitely often "p"

\[ \mathcal{E}(\nu x. \mu y. (p \land \diamond x) \lor \diamond y, (T, s_1)) : \]
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

Games for the modal $\mu$-calculus

\begin{align*}
&\langle \diamond x, s_4 \rangle \\
&\langle \nu x.\mu y.(p \land \diamond x) \lor \diamond y, s_4 \rangle \\
&\langle \diamond y, s_4 \rangle \\
&\langle \mu y.(p \land \diamond x) \lor \diamond y, s_1 \rangle
\end{align*}
Game-theoretical version of the “fundamental theorem”

Theorem [Streett Emerson 89]
\[ s \in \|\varphi\|_\mathcal{T} \iff \text{V has a winning strategy in } \mathcal{E}(\varphi, (\mathcal{T}, s)). \]
Game Formulae
For all \( n \geq 1 \) we define the \( \Sigma^\mu_n \) Game formula \( W_{\Sigma^\mu_n} \) and the \( \Pi^\mu_n \) Game formula \( W_{\Pi^\mu_n} \) such that (\( n \) even):

\[
W_{\Sigma^\mu_n} \equiv \mu x_{n+1} . \nu x_n . \ldots \nu / \mu x_2 \left( \bigvee_{i=2}^{n+1} (d_i \land \Diamond x_i) \lor \bigvee_{i=2}^{n+1} (c_i \land \Box x_i) \right)
\]

\[
W_{\Pi^\mu_n} \equiv \nu x_{n+2} . \mu x_{n+1} . \ldots \mu / \nu x_3 \left( \bigvee_{i=3}^{n+2} (d_i \land \Diamond x_i) \lor \bigvee_{i=3}^{n+2} (c_i \land \Box x_i) \right)
\]
Game Formulae
For all $n \geq 1$ we define the $\Sigma_n^\mu$ Game formula $W_{\Sigma_n^\mu}$ and the $\Pi_n^\mu$ Game formula $W_{\Pi_n^\mu}$ such that ($n$ even):

$W_{\Sigma_n^\mu} \equiv \mu x_{n+1} . \nu x_n . . . . \nu / \mu x_2 (\bigvee_{i=2}^{n+1} (d_i \land \Diamond x_i) \lor \bigvee_{i=2}^{n+1} (c_i \land \Box x_i))$

$W_{\Pi_n^\mu} \equiv \nu x_{n+2} . \mu x_{n+1} . . . . \mu / \nu x_3 (\bigvee_{i=3}^{n+2} (d_i \land \Diamond x_i) \lor \bigvee_{i=3}^{n+2} (c_i \land \Box x_i))$

$W_{\Sigma_n^\mu} \in \Sigma_n^\mu$ and $W_{\Pi_n^\mu} \in \Pi_n^\mu$. 
Theorem [Emerson, Jutla (91), Walukiewicz (00)]

Let $\varphi$ be a $\Pi_n^\mu$-formula and $(T, s)$ be a pointed transition system. Player $V$ has a winning strategy for $E(\varphi, (T, s))$ if and only if $T(E(\varphi, (T, s))) \in \| W_{\Pi_n^\mu} \|$; similarly for $\Sigma_n^\mu$-formulae.
Theorem [Emerson, Jutla (91), Walukiewicz (00)]

Let $\varphi$ be a $\Pi^\mu_n$-formula and $(T, s)$ be a pointed transition system. Player $V$ has a winning strategy for $E(\varphi, (T, s))$ if and only if $T(E(\varphi, (T, s))) \in \| W_{\Pi^\mu_n} \|$; similarly for $\Sigma^\mu_n$-formulae.

Corollary

$$(T, s) \in \| \varphi \| \iff T(E(\varphi, (T, s))) \in \| W_{\Pi^\mu_n} \|;$$

similarly for $\Sigma^\mu_n$-formulae.
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on Reflexive Transition Systems
Construct $E^r(\varphi, (\mathcal{T}, s))$ by making the "moves" relation $E$ reflexive and adapting $\Omega$ to $\Omega^r$:

$$\Omega^r(\langle \psi, s \rangle) = \Omega(\langle \psi, s \rangle) \quad \psi \equiv \eta x. \alpha$$

$$\Omega^r(\langle \psi, s \rangle) = \begin{cases} 0 & \text{if } \langle \psi, s \rangle \in V_1 \\ 1 & \text{if } \langle \psi, s \rangle \in V_0 \end{cases} \quad \psi \not\equiv \eta x. \alpha$$

**Lemma**

Player $V$ has a winning strategy for $E^r(\varphi, (\mathcal{T}, s))$ iff Player $V$ has a winning strategy for $E(\varphi, (\mathcal{T}, s))$. 
For all $n \geq 0$ we define the $\Sigma_n^{\mu}$ Walukiewicz formula $W_{\Sigma_n^{\mu}}$ and the $\Pi_n^{\mu}$ Walukiewicz formula $W_{\Pi_n^{\mu}}$ such that ($n$ even):

$$W_{r_{\Sigma_n^{\mu}}} \equiv \mu x_{n+1} . \nu x_n \ldots \nu / \mu x_0 (\bigvee_{i=0}^{n+1} (d_i \land \Diamond x_i) \lor \bigvee_{i=0}^{n+1} (c_i \land \Box x_i))$$
Reflexive Game formula

For all $n \geq 0$ we define the $\Sigma^\mu_n$ Walukiewicz formula $W_{\Sigma^\mu_n}$ and the $\Pi^\mu_n$ Walukiewicz formula $W_{\Pi^\mu_n}$ such that ($n$ even):

\[ W_{\Sigma^\mu_n}^r := \mu x_{n+1}.\nu x_n \ldots \nu / \mu x_0 \left( \bigvee_{i=0}^{n+1} (d_i \land \lozenge x_i) \lor \bigvee_{i=0}^{n+1} (c_i \land \Box x_i) \right) \]

$W_{\Sigma^\mu_n}^r \in \Sigma^\mu_{n+2}$ and $W_{\Pi^\mu_n}^r \in \Pi^\mu_{n+2}$. 
Proposition

Let $(\mathcal{T}, s)$ be an arbitrary pointed transition system. For all $\varphi \in \Pi_n^\mu$ we have that:

$$\mathcal{T}(\mathcal{E}^r(\varphi, (\mathcal{T}, s))) \in \| W_{\Pi_n^\mu}^r \|$$

if and only if $(\mathcal{T}, s) \in \| \varphi \|$.  

and analogously for $W_{\Sigma_n^\mu}^r$. 
Theorem
For all natural numbers $n \in \mathbb{N}$ we have that

$$\Sigma_n^{Tr} \subset \Sigma_{n+1}^{Tr} \quad \text{and} \quad \Pi_n^{Tr} \subset \Pi_{n+1}^{Tr}.$$
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on Reflexive Transition Systems

Theorem
For all natural numbers $n \in \mathbb{N}$ we have that

$$\Sigma^r_n \subsetneq \Sigma^r_{n+1} \quad \text{and} \quad \Pi^r_n \subsetneq \Pi^r_{n+1}.$$ 

Proof

Else for all $k$

$$\Sigma^r_n = \Sigma^r_{n+k} = \Pi^r_n = \Pi^r_{n+k}$$

and $\|W^r_{\Sigma^r_n}\| \in \Pi^r_n \text{ or } \|\neg W^r_{\Sigma^r_n}\| \in \Sigma^r_n$. 
Theorem
For all natural numbers \( n \in \mathbb{N} \) we have that

\[
\Sigma^T_{n} \subset \Sigma^T_{n+1} \quad \text{and} \quad \Pi^T_{n} \subset \Pi^T_{n+1}.
\]

Proof
Else for all \( k \)

\[
\Sigma^T_{n} = \Sigma^T_{n+k} = \Pi^T_{n} = \Pi^T_{n+k}
\]

and \( W_{\Sigma^T_{n} \mu}^r \in \Pi^T_{n} \) or \( \neg W_{\Sigma^T_{n} \mu}^r \in \Sigma^T_{n} \). Construct \((T^F, s^F)\) such that \( T(E^r(\neg W_{\Sigma^T_{n} \mu}^r, (T^F, s^F))) = (T^F, s^F)\).
Theorem
For all natural numbers \( n \in \mathbb{N} \) we have that

\[
\Sigma_{n}^{Tr} \not\subseteq \Sigma_{n+1}^{Tr} \quad \text{and} \quad \Pi_{n}^{Tr} \not\subseteq \Pi_{n+1}^{Tr}.
\]

Proof
Else for all \( k \)

\[
\Sigma_{n}^{Tr} = \Sigma_{n+k}^{Tr} = \Pi_{n}^{Tr} = \Pi_{n+k}^{Tr}
\]

and \( \| W_{\Sigma_{n}^{\mu}}^{r} \|^{r} \in \Pi_{n}^{Tr} \) or \( \| \neg W_{\Sigma_{n}^{\mu}}^{r} \|^{r} \in \Sigma_{n}^{Tr} \). Construct \((T^{F}, s^{F})\) such that \( T(\mathcal{E}^{r}(\neg W_{\Sigma_{n}^{\mu}}^{r}, (T^{F}, s^{F}))) = (T^{F}, s^{F}) \). We have

\[
(T^{F}, s^{F}) \in \| \neg W_{\Sigma_{n}^{\mu}}^{r} \| \quad \text{iff} \quad T(\mathcal{E}^{r}(\neg W_{\Sigma_{n}^{\mu}}^{r}, (T^{F}, s^{F}))) \in \| W_{\Sigma_{n}^{\mu}}^{r} \|.
\]
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on transitive and symmetric Transition Systems
Lemma

Let $\mathcal{T}$ be a transitive transition system and let $s' \in \text{scc}(s)$. For all $\mu$-formulae $\varphi$ we have that

$$s \in \parallel \Delta \varphi \parallel_{\mathcal{T}} \text{ iff } s' \in \parallel \Delta \varphi \parallel_{\mathcal{T}}$$

where $\Delta \in \{\Box, \Diamond\}$. 

The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on transitive and symmetric Transition Systems

Theorem

Let $T$ be a transitive and symmetric transition system. We have that

$$\|\nu x. \varphi(x)\|_T = \|\varphi(\varphi(T))\|_T.$$
The syntactical translation \((\cdot)^t : \mathcal{L}_\mu \rightarrow \mathcal{L}_M\) is defined as:

- \(\ldots\)
- \((\mu x.\varphi)^t = (\varphi(\varphi(\bot)))^t\)
- \((\nu x.\varphi)^t = (\varphi(\varphi(\top)))^t\)

**Corollary**

On transitive and symmetric (and reflexive) transition systems we have that

\[\|\varphi\|_\mathcal{T} = \|\varphi^t\|_\mathcal{T}.\]
The modal \(\mu\)-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on transitive Transition Systems
Lemma
Let $\mathcal{T}$ be a transitive transition system and let $s, s'$ be two states such that $s \xrightarrow{\mathcal{T}} s'$. For all $\mu$-formulae $\varphi$ we have that

$$s \in \llbracket\Box \varphi\rrbracket_{\mathcal{T}} \implies s' \in \llbracket\Box \varphi\rrbracket_{\mathcal{T}}$$

and

$$s' \in \llbracket\Diamond \varphi\rrbracket_{\mathcal{T}} \implies s \in \llbracket\Diamond \varphi\rrbracket_{\mathcal{T}}.$$ 

Theorem
Let $\mathcal{T}$ be a transitive transition system and let $\nu x. \varphi(x)$ be a formula such that $x$ is in the scope of a $\Box$ modality. We have that

$$\llbracket\nu x. \varphi(x)\rrbracket_{\mathcal{T}} = \llbracket\varphi(\varphi(\top))\rrbracket_{\mathcal{T}}.$$
The modal $\mu$-calculus Hierarchy on Restricted Classes of Transition Systems

The Hierarchy on transitive Transition Systems

$\tau : \mathcal{L}_\mu \rightarrow \mathcal{L}_\mu$ is defined as:

- $\tau(\mu x.\varphi) = \tau(\varphi(\varphi(\bot)))$, $x$ is in the scope of a $\Diamond$ in $\varphi$
- $\tau(\mu x.\varphi) = \mu x.\tau(\varphi)$, $x$ is not in the scope of a $\Diamond$ in $\varphi$
- $\tau(\nu x.\varphi) = \tau(\varphi(\varphi(\top)))$, $x$ is in the scope of a $\Box$ in $\varphi$
- $\tau(\nu x.\varphi) = \nu x.\tau(\varphi)$, $x$ is not in the scope of a $\Box$ in $\varphi$
\( \tau : \mathcal{L}_\mu \rightarrow \mathcal{L}_\mu \) is defined as:

- \( \tau(\mu x.\varphi) = \tau(\varphi(\varphi(\bot))) \), \( x \) is in the scope of a \( \Diamond \) in \( \varphi \)
- \( \tau(\mu x.\varphi) = \mu x.\tau(\varphi) \), \( x \) is not in the scope of a \( \Diamond \) in \( \varphi \)
- \( \tau(\nu x.\varphi) = \tau(\varphi(\varphi(\top))) \), \( x \) is in the scope of a \( \Box \) in \( \varphi \)
- \( \tau(\nu x.\varphi) = \nu x.\tau(\varphi) \), \( x \) is not in the scope of a \( \Box \) in \( \varphi \)

**Corollary**

On transitive transition systems we have that

\[ \|\varphi\|_\mathcal{I} = \|\tau(\varphi)\|_\mathcal{I}. \]
Notation and Definitions

Let $\varphi(x_1, \ldots, x_n)$ be a formula by $\varphi^{x_i}$ we denote the formula obtained by cutting all branches except $x_i$.

$$(\nu x. \mu y. \mu z. (\square x \land \Diamond y) \lor (\Diamond z \land p))^x \equiv \nu x. \square x \lor p$$
Notation and Definitions

- Let $\varphi(x_1, \ldots, x_n)$ be a formula by $\varphi^{x_i}$ we denote the formula obtained by cutting all branches except $x_i$.

$$ (\nu x.\mu y.\mu z.(\Box x \land \Diamond y) \lor (\Diamond z \land p))^x \equiv \nu x.\Box x \lor p $$

- For all set of variables $X$ the formula $\varphi^{\text{free}(X)}$ is the formula obtained from $\varphi$ by eliminating all quantifiers binding a variable $x \in X$ but leaving the previously bound variable $x$ as a free occurrence.

$$ (\nu x.\mu y.\mu z.(\Box x \land \Diamond y) \lor (\Diamond z \land p))^{\text{free}(x,y,z)} \equiv (\Box x \land \Diamond y) \lor (\Diamond z \land p) $$
For each sequence of $\langle x_1, \ldots, x_k \rangle$ with $x_j \in \text{bound}(\varphi)$, we define the formula $\varphi^{\langle x_1, \ldots, x_k \rangle}$ as follows:

$$\varphi^{\langle x_1 \rangle} : \equiv \varphi^{x_1}$$

and

$$\varphi^{\langle x_1, \ldots, x_k, x_{k+1} \rangle} : \equiv \varphi^{\langle x_1, \ldots, x_k \rangle}[x_k / \varphi^{x_{k+1}}].$$
Let \( \varphi \) be a \( \mu \)-formula and \( X, Y \subseteq \text{bound}(\varphi) \). \( \text{Path}^{X \rightarrow Y}(\varphi) \) is the smallest set such that for all \( x \in X \)

\[
\{ \langle x, y \rangle ; \ y \in \text{free}(\varphi_x) \text{ and } y \in Y \} \subseteq \text{Path}^{X \rightarrow Y}(\varphi)
\]

and such that if \( \langle x_1, \ldots, x_m, y \rangle \in \text{Path}(\varphi) \), if \( x' \in X \), if \( x' \not\in \{x_1, \ldots, x_m\} \) and if \( x_1 \in \text{free}(\varphi_{x'}) \) then

\[
\langle x', x_1, \ldots, x_n, y \rangle \in \text{Path}^{X \rightarrow Y}(\varphi).
\]
Let \( \varphi \) be a \( \mu \)-formula and \( X, Y \subset \text{bound}(\varphi) \). Path\(^{X \rightarrow Y}(\varphi)\) is the smallest set such that for all \( x \in X \)

\[
\{\langle x, y \rangle ; \ y \in \text{free}(\varphi_x) \text{ and } y \in Y \} \subseteq \text{Path}\(^{X \rightarrow Y}(\varphi)\)
\]

and such that if \( \langle x_1, \ldots, x_m, y \rangle \in \text{Path}(\varphi) \), if \( x' \in X \), if \( x' \not\in \{x_1, \ldots, x_m\} \) and if \( x_1 \in \text{free}(\varphi_{x'}) \) then

\[
\langle x', x_1, \ldots, x_n, y \rangle \in \text{Path}\(^{X \rightarrow Y}(\varphi)\).
\]

For all \( x \in X \) we define

\[
\text{Path}\(^{x \rightarrow Y}(\varphi) = \{\langle x_1, \ldots, x_k, y \rangle \in \text{Path}\(^{X \rightarrow Y}(\varphi) \text{ ; } x_1 \equiv x \}\.
\]
Let $\varphi$ be a $\mu$-formula and $X, Y \subseteq \text{bound}(\varphi)$. $\text{Path}^{X\rightarrow Y}(\varphi)$ is the smallest set such that for all $x \in X$

\[
\{(x, y) ; y \in \text{free}(\varphi_x) \text{ and } y \in Y\} \subseteq \text{Path}^{X\rightarrow Y}(\varphi)
\]

and such that if $\langle x_1, \ldots, x_m, y \rangle \in \text{Path}(\varphi)$, if $x' \in X$, if $x' \not\in \{x_1, \ldots, x_m\}$ and if $x_1 \in \text{free}(\varphi_{x'})$ then

\[
\langle x', x_1, \ldots, x_n, y \rangle \in \text{Path}^{X\rightarrow Y}(\varphi).
\]

For all $x \in X$ we define

\[
\text{Path}^{x\rightarrow Y}(\varphi) = \{(x_1, \ldots, x_k, y) \in \text{Path}^{X\rightarrow Y}(\varphi) ; x_1 \equiv x\}.
\]

The formula $\varphi^{x_i \rightarrow Y}$ is defined such that

\[
\varphi^{x_i \rightarrow Y} \equiv \bigvee_{s \in \text{Path}^{x_i \rightarrow Y}} \varphi^s_{x_i}.
\]
The unfolding of $X$ in $\psi$ as subformula of $\varphi$, $\text{unf}_\varphi^X(\psi)$, is the formula defined recursively such that

$$\text{unf}_{\varphi}^{\{x_1\}}(\psi) \equiv \psi[x/\varphi_x]$$

and such that if $X = \{x_1, \ldots, x_n\}$ then

$$\text{unf}_\varphi^X(\psi) \equiv \psi[x_1/\text{unf}_\varphi^{X^{-1}}(\varphi_{x_1}), \ldots, x_n/\text{unf}_\varphi^{X^{-n}}(\varphi_{x_n})]$$

where $X^{-i} = \{x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}$. 
The Translation

\( \varphi \in \Sigma_2^\mu \) with \( \{x_1, \ldots, x_n\} = X \) all \( \mu \)-variables and \( \{y_1, \ldots, y_m\} = Y \) all \( \nu \)-variables. We define \( \rho(\varphi) \in \Delta_2^\mu \) as

\[
\varphi^{\text{free}(X)}[x_1/\varphi^{x_1 \rightarrow Y} \lor \text{unf}^X_{\varphi\rightarrow Y} (\varphi_{x_1}^{x_1 \rightarrow Y}), \ldots, x_n/\varphi^{x_n \rightarrow Y} \lor \text{unf}^X_{\varphi\rightarrow Y} (\varphi_{x_n}^{x_n \rightarrow Y})].
\]
Lemma
Let $\mathcal{T}$ be a transitive transition system, and let $\varphi \in \Sigma^\mu_2$ such that all $\nu$-variables (resp. $\mu$-variables) $x$ are in the scope of only $\Diamond$ (resp. $\Box$). Then we have

$$\|\varphi\|_T = \|\rho(\varphi)\|_T$$
Lemma
Let $T$ be a transitive transition system, and let $\varphi \in \Sigma^\mu_2$ such that all $\nu$-variables (resp. $\mu$-variables) $x$ are in the scope of only $\Diamond$ (resp. $\Box$). Then we have

$$\|\varphi\|_T = \|\rho(\varphi)\|_T$$

Proof
Show the existence of a normal form for winning plays for player $V$ of $\mathcal{E}(\varphi, (T, s))$ and show that these plays are winning for $V$ in $\mathcal{E}(\rho(\varphi), (T, s))$; and vice versa.
\( R : \mathcal{L}_\mu \rightarrow \Delta^\mu_2 \) is defined as

- \( R(\mu x. \varphi) = \rho(\text{nf}(\mu x. (R(\varphi)))) \)
- \( R(\nu x. \varphi) = \neg(R(\mu x. \neg \varphi[x/\neg x])) \)
Theorem
For all $\varphi \in \mathcal{L}_\mu$ and all transitive transition systems $\mathcal{T}$ we have that

$$\|\varphi\|_\mathcal{T} = \|R(\tau(\varphi))\|_\mathcal{T}.$$
Thank you!
Thank you!

Questions or Remarks?