Topological, simplicial and categorical joins

Rudolf Fritsch and Marek Golasiński

Abstract. We examine different constructions for the join of two topological spaces, simplicial sets and small categories. We give a systematic account of this and try to clarify the interrelations between the various constructions as much as possible. In particular, we compare two standard versions of the join of simplicial sets proving results which apparently are not yet in the literature.

Introduction. In several categories the join of two objects is an important construction and in each of these cases there are several versions of this construction. We give a systematic account of this and try to clarify the interrelations between the various constructions as much as possible. In doing so we recall some known results, but some others seem to be new. We did not find them in the literature.

The join $X * Y$ of two topological spaces X and Y appears in many homotopical constructions. In the light of [7] some difficult results are easy consequences of "join theorems". The Hopf construction H assigns to each map $f : X \times Y \to Z$ a map $H(f) : X * Y \to \Sigma Z$ to the suspension ΣZ of the space Z and, in particular, for well-pointed spaces X, Y, to the canonical map $X \times Y \to X \wedge Y$, a homotopy equivalence $X * Y \xrightarrow{\simeq} \Sigma(X \wedge Y)$. The homotopy category of topological spaces can be described by simplicial sets and small categories, and all of them are connected by appropriate functors (see e.g., [3], [10], [17] or [23]). In [2] and [13] a notion of a join $\mathbb{C} * \mathbb{C}'$ of two small categories $\mathbb{C}$ and $\mathbb{C}'$ is considered and its homology groups are expressed by those of $\mathbb{C}$ and $\mathbb{C}'$.

In Section 1 we first examine two notions $X * Y$ and $CX \times Y \cup X \times CY$ of the join of two topological spaces X, Y and prove that the spaces obtained are homotopy equivalent. We also draw the reader's attention to a similar construction due to J.

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Milnor [18] (see also [5, pp. 159-160]) which gives the same homotopy type too [21, Theorem 1].

In Section 2 we consider joins in the category of k-spaces. Here the two constructions from Section 1 give canonically homeomorphic results which we both denote $X *_k Y$. Since k-ification does not change weak homotopy type this join and all three versions of the topological join in Section 1 have the same weak homotopy type.

In Section 3 we present some facts on the join of finite ordinals and establish interrelations between three notions: $\mathbb{X} *_w \mathbb{Y}$, $\mathbb{X} * \mathbb{Y}$ and $\operatorname{hocolim} F$ of the join of simplicial sets $\mathbb{X}$ and $\mathbb{Y}$ based on the double mapping cylinder, the join of finite ordinals and simplicial complexes (see e.g., [8, p. 74], [22, p. 109]), and the homotopy colimit (cf. [6]) of a functor F associated to simplicial sets $\mathbb{X}$ and $\mathbb{Y}$, respectively. First, we observe that there is a canonical homeomorphism $|\mathbb{X} *_w \mathbb{Y}| \xrightarrow{\cong} |\mathbb{X}| *_k |\mathbb{Y}|$ because the geometric realization functor $|-|$ to k-spaces preserves colimits and finite products. Second, in Theorem 3.3, we establish a natural homeomorphism $|\mathbb{X} *_w \mathbb{Y}| \xrightarrow{\cong} |\mathbb{X} * \mathbb{Y}|$ which is homotopic to the geometric realization of a simplicial map $\mathbb{X} *_w \mathbb{Y} \to \mathbb{X} * \mathbb{Y}$. It seems that a discussion of this relation is not yet in the literature.

In the final section, Section 4, we point out that the canonical join construction in the category of small categories (see [13]) is compatible with the embedding into the category of simplicial sets via the nerve functor and connect it to the so-called Grothendieck construction. Consequently, the result presented in [2] and [13] on the homology groups of $\mathbb{C} * \mathbb{C}'$ might be derived directly.

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1. Topological joins. Let X and Y be topological spaces and I the unit interval of the real numbers. Then the *join* $X * Y$ of X and Y is defined to be the double mapping cylinder of the diagram

$$X \overset{p_X}{\longleftarrow} X \times Y \overset{p_Y}{\longrightarrow} Y,$$

where p_X and p_Y are the projection maps (cf. e.g., [4, p. 486], [12, p. 334], [14, p. 114], [15, p. 213] or [19, p. 186]). Thus $X * Y$ is the space obtained from the disjoint union (coproduct)

$$X \sqcup X \times Y \times I \sqcup Y$$

by the identifications $x \sim (x, y, 0)$ and $(x, y, 1) \sim y$ for all $x \in X$ and $y \in Y$. We shall write $< x, y, t >\in X * Y$ for the class of $(x, y, t) \in X \times Y \times I$. Every element of $X * Y$ has this form except if one of X, Y is empty and the other not. In the latter case we have $X * \emptyset = X$, $\emptyset * Y = Y$.

There is an alternative construction of joins based on the cone concept. The cone CX over a space X can be defined as $\mathrm{pt} * X$, where pt is a one point space. Directly it can be obtained from $\mathrm{pt} \sqcup X \times I$ by identifying pt with all of $X \times 0$. We shall write $[x, t] \in CX$ for the class of $(x, t) \in X \times I$.

It is not hard to check that $X \times CY \cup CX \times Y$ has the same topology if considered as a subspace of $CX \times CY$ or as the colimit of the diagram

$$X \times CY \hookleftarrow X \times Y \hookrightarrow CX \times Y.$$

Some authors define this space as the join of X and Y (cf. e.g., [14, p. 114], [19, p. 186] or [27, p. 482]).

There is a canonical continuous bijection

$$\phi : X * Y \longrightarrow CX \times Y \cup X \times CY$$

which sends $< x, y, t >\in X * Y$ to $(x, [y, 2t]) \in X \times CY$ if $t \leq \frac{1}{2}$ and to $([x, 2t-2], y) \in CX \times Y$ if $t \geq \frac{1}{2}$. This map becomes a homeomorphism if the topology of $X \times CY$ is replaced by the quotient topology of $X \sqcup X \times Y \times I$ under the canonical projection and similarly for $CX \times Y$. The quotient topology is the same as the original one if X and Y are locally compact (in the sense of [5, p. 84]) but not in general (cf. Example 1.2 below). On the other hand, we always have

Proposition 1.1. *The map ϕ is a homotopy equivalence.*

The proof works by a well known method of "shrinking and stretching the line" used e.g., in [20]. We obtain a homotopy inverse of ϕ by applying ϕ^{-1} after sending $(x,[y,t]) \in X \times CY$ to $(x,[y,0])$ if $t \leq \frac{1}{2}$ and to $(x,[y,2t-2])$ if $t \geq \frac{1}{2}$, and analogously for the part $CX \times Y$.

To show that the described map ϕ fails to be a homeomorphism in general (opposite to that what is stated in some places) analyse an example based on [9, Example 2.4.20].

Example 1.2. Let $\mathbb{R}$ be the space of real numbers and $\mathbb{N}$ the natural numbers. Take $X = \mathbb{R}\backslash\{\frac{1}{j};\, j \in \mathbb{N}\}$, $Y = \mathbb{N}$ with subspace topologies and $\Theta : Y \times I \to CY = Y \times I/Y \times 0$ the canonical quotient map. We show that the topology of $X \times CY$ is not induced by the canonical map $\Theta' = X \times \Theta : X \times Y \times I \to X \times CY$.

Consider the subset $\bar{F} = \{(x,j,t); |\frac{1}{j} - x| \leq t\}$ of $\mathbb{R} \times Y \times I$ and the intersection $F = \bar{F} \cap (X \times Y \times I)$. The set $\bar{F}$ is a discrete union of closed triangles, thus it is closed and consequently, F is closed in the subspace topology; the set F may be seen as a union of "porous" triangles with cut peaks. Because $F \cap (X \times Y \times \{0\}) = \emptyset$, the set F is also saturated and thus its image $\Theta'(F)$ is closed in the quotient topology. But it is not closed in the product topology since, as one easily checks, the image of the point $(0,1,0) \notin F$ is in its closure.

There is still another variation of the join construction, due to J. Milnor [18] (see also [5, pp. 159-160]). Given non-empty spaces X, Y, form $X \bar{*} Y$ as the quotient set of the set of all quadruples (r,x,s,y) with r, $s \in [0,1]$, $r+s=1$, $x \in X$, $y \in Y$. The quotient is formed by identifying $(0,x,1,y)$ and $(0,x',1,y)$ as well as $(1,x,0,y)$ and $(1,x,0,y')$. The topology is chosen as the initial one for the coordinate maps as far as they are defined. For compact spaces X, Y the canonical map

$$X * Y \longrightarrow X \bar{*} Y$$

is a homeomorphism [5, p. 166, Corollary 1] and [25, Proposition 3.11]. Thus, in this case, all three join constructions end up with the same space (up to homeomorphism). In general, this map is again a homotopy equivalence as a result of the method of "shrinking and stretching the line" [21, Theorem 1].

2. Joins in the category of k-spaces.

A topological space X is said to be a *k-space* if it has the final topology with respect to all maps $C \to X$ for all compact Hausdorff spaces C. The property of being a k-space is preserved by closed subspaces and quotients. All metric spaces and all CW-spaces are k-spaces. For arbitrary spaces X there is its *k-ification* kX (see e.g., [5, p. 174] and [11, p. 242]), the space having the same underlying set as X, but with the topology given by taking as closed sets the compactly closed sets with respect to the topology of X. Note that the identity map $kX \to X$ is continuous and a weak homotopy equivalence. Moreover, if Y is a k-space then a map $f : Y \to X$ is continuous if and only if the map $f : Y \to kX$ is continuous. If X and Y are k-spaces then their product $X \times_k Y$ (in the category of k-spaces) is given by $k(X \times_c Y)$, where $X \times_c Y$ is the Cartesian product (endowed with the product topology). If at least one of the spaces X or Y is locally compact then $X \times_k Y = X \times_c Y$.

In the the category of k-spaces quotients and finite products commute. This implies that the double mapping cylinder of

$$X \xleftarrow{p_X} X \times_k Y \xrightarrow{p_Y} Y$$

is canonically homeomorphic to the pushout of

$$CX \times_k Y \longleftarrow X \times_k Y \longrightarrow X \times_k CY.$$

For both of the resulting spaces we use the notation $X *_k Y$ and call it the *k-join*. From the fact that each of $X \times Y$, $CX \times Y$ and $X \times CY$ is a closed subspace of $CX \times CY$ one easily sees that $X *_k Y$ is a closed subspace of $CX *_k CY$ and coincides with $k(CX \times Y \cup X \times CY)$. Since the functor k does not change weak homotopy type $X *_k Y$ has the same weak homotopy type as $CX \times Y \cup X \times CY$ and hence also as $X * Y$ and $X \bar{*} Y$.

For CW-complexes X and Y, the spaces $X * Y$, $X \bar{*} Y$ and $CX \times Y \cup X \times CY$ are not CW-complexes in general, but the space $X *_k Y$ has a canonical structure of a CW-complex.

3. Simplicial joins.

In this section we use the terminology and notations from [11, Chap. 4]; in addition to the developments in this reference the idea of a unique empty simplex has to be taken into account for our purposes.

If $K_1 = (V_1, S_1)$ and $K_2 = (V_2, S_2)$ are simplicial complexes then their *join* $K_1 * K_2$ (see e.g., [8, p. 74] and [22, p. 109]) is the simplicial complex defined by

$$(V_1 \sqcup V_2, \{s_1 \sqcup s_2; s_1 \in S_1, s_2 \in S_2\}).$$

A simplicial complex, after ordering its vertices, can be interpreted as a simplicial set. In this sense simplices correspond to finite ordinals seen as standard simplices within the category of simplicial sets. More precisely, the simplicial complex formed by the geometric n-simplex Δ^n and its faces is described by the ordinal $[n] = \{0, 1, \ldots, n\}$ and all its subsets. The empty simplex to which one usually assigns the dimension -1 corresponds to the empty set, denoted by $[-1]$ if considered as a finite ordinal. The category $\mathcal{O}$ of finite ordinals has as objects the finite ordinals $[-1]$, $[0]$, $[1]$, $\ldots$ and as morphisms the weakly increasing functions, also called *operators*.

Define the *join functor*

$$* : \mathcal{O} \times \mathcal{O} \longrightarrow \mathcal{O}$$

for the category $\mathcal{O}$ as follows. For objects $[m]$, $[n]$ take

$$[m] * [n] = [m + n + 1]$$

and for $\beta : [m'] \to [m]$, $\gamma : [n'] \to [n]$ the operator

$$\beta * \gamma : [m' + n' + 1] \longrightarrow [m + n + 1]$$

is given by

$$(\beta * \gamma)(i) = \begin{cases} \beta(i), & 0 \leq i \leq m'; \\ \gamma(i - m' - 1) + m + 1, & m' + 1 \leq i \leq m' + n' + 1. \end{cases}$$

This join for finite ordinals is extended to the category of simplicial sets being the cocompletion of the category $\mathcal{O}$. To this aim, a fact on the join of finite ordinals is needed which can be easily checked.

Lemma 3.1. *Given k, m, $n \in \{-1, 0, 1, \ldots\}$ and an operator $\alpha : [k] \to [m+n+1]$ there is exactly one pair of numbers $m', n' \in \{-1, 0, 1, \ldots\}$ and one pair of operators $\beta : [m'] \to [m]$, $\gamma : [n'] \to [n]$ such that $m' + n' + 1 = k$ and $\beta * \gamma = \alpha$.*

□

A simplicial set $\mathbb{X}$ is a cofunctor from $\mathcal{O}$ to the category $\mathbb{S}et$ of sets which sends $[-1]$ (the initial object of the category $\mathcal{O}$) into a singleton $\mathbb{X}_{-1}$ (a terminal object of the category $\mathbb{S}et$); the category of simplicial sets is denoted by $\mathbb{S}i\mathbb{S}ets$. The Yoneda embedding $\Delta : \mathcal{O} \to \mathbb{S}i\mathbb{S}ets$ sends each object $[n]$ into the cofunctor $\Delta[n]$ which it represents. Define the *join* of simplicial sets as the left Kan extension of the composed functor

$$\mathcal{O} \times \mathcal{O} \xrightarrow{*} \mathcal{O} \xrightarrow{\Delta} \mathbb{S}i\mathbb{S}ets$$

along the embedding $\Delta \times \Delta : \mathcal{O} \times \mathcal{O} \longrightarrow \mathbb{S}i\mathbb{S}ets \times \mathbb{S}i\mathbb{S}ets$. Explicitly, the join $\mathbb{X} * \mathbb{Y}$ of two simplicial sets $\mathbb{X}$, $\mathbb{Y}$ is the quotient of the simplicial set

$$\coprod_{m,n=-1}^{\infty} \mathbb{X}_m \times \mathbb{Y}_n \times \Delta[m+n+1] \tag{1}$$

under the equivalence relation $\sim$ generated by $(x\beta, y\gamma, \alpha) \sim (x, y, (\beta * \gamma)\alpha)$. The simplicial operators of $\mathbb{X} * \mathbb{Y}$ just act on the third factor in this description.

Lemma 3.1 shows that there is a canonical bijection

$$\coprod_{m=-1}^{k} \mathbb{X}_m \times \mathbb{Y}_{k-m-1} \longrightarrow (\mathbb{X} * \mathbb{Y})_k \tag{2}$$

defined by sending $(x, y) \in \mathbb{X}_m \times \mathbb{Y}_{k-m-1}$ to the equivalence class of $(x, y, \mathrm{id}_{[k]})$. The description of the simplices of $\mathbb{X} * \mathbb{Y}$ provided by the source of this bijection is obviously analogous to the definition of the join of two simplicial complexes given at the beginning of this section.

The first join construction for topological spaces suggests another one for simplicial sets. The *weak join* $\mathbb{X} *_w \mathbb{Y}$ of simplicial sets $\mathbb{X}$ and $\mathbb{Y}$ is defined to be the double mapping cylinder of the diagram

$$\mathbb{X} \xleftarrow{\pi_{\mathbb{X}}} \mathbb{X} \times \mathbb{Y} \xrightarrow{\pi_{\mathbb{Y}}} \mathbb{Y},$$

where $\pi_{\mathbb{X}}$ and $\pi_{\mathbb{Y}}$ are the projection maps. Thus it is a quotient of the coproduct $\mathbb{X} \sqcup \mathbb{X} \times \mathbb{Y} \times \Delta[1] \sqcup \mathbb{Y}$ with appropriate identifications. We shall write $< x, y, \tau > \in \mathbb{X} *_w \mathbb{Y}$ for the class of $(x, y, \tau) \in \mathbb{X} \times \mathbb{Y} \times \Delta[1]$. Every element of $\mathbb{X} *_w \mathbb{Y}$ has this form except if one of $\mathbb{X}$ or $\mathbb{Y}$ is isomorphic to $\Delta[-1]$ (i.e., empty in all dimensions ≥ 0) and the other not. We will have to pay attention to $\Delta[-1]$.

Geometric realization $|-|$ is a functor from simplicial sets to k-spaces which commutes with coproducts, finite products and identifications, and it sends $\Delta[1]$ to the unit interval. Hence we have

Proposition 3.2. *For all simplicial sets $\mathbb{X}$ and $\mathbb{Y}$, there is a canonical natural homeomorphism*

$$|\mathbb{X} *_w \mathbb{Y}| \xrightarrow{\cong} |\mathbb{X}| *_k |\mathbb{Y}|.$$

□

Note that the weak join construction defines a bifunctor into the category $\mathbb{S}i\mathbb{S}ets$ of simplicial sets. The connection between both joins of simplicial sets $\mathbb{X}$ and $\mathbb{Y}$ is given in the following

Theorem 3.3. (a) *There exists one and only one natural transformation*

$$\varphi : \mathbb{X} *_w \mathbb{Y} \longrightarrow \mathbb{X} * \mathbb{Y}$$

from the weak join to the join of simplicial sets.

(b) *The geometric realizations* $|\varphi|$ *of the simplicial maps* φ *are naturally homotopic to natural homeomorphisms*

$$\Phi : |\mathbb{X}| *_k |\mathbb{Y}| = |\mathbb{X} *_w \mathbb{Y}| \longrightarrow |\mathbb{X} * \mathbb{Y}| \,.$$

Proof. (a) It is easy to see that the weak join of simplicial sets commutes with colimits of connected diagrams (i.e., functors whose source category is connected). Every simplicial set $\mathbb{X}$ may be canonically represented as the colimit of standard simplices $\Delta[m]$ corresponding to the elements of $\mathbb{X}_m$ [11, Lemma 4.2.1]. In our setting the integer m takes all values from -1 on, and this implies that the source category for the colimit is connected. It even has an initial object, namely the unique element of $\mathbb{X}_{-1}$. Therefore, the weak join is the left Kan extension of its restriction to $\mathcal{O} \times \mathcal{O}$ along $\Delta \times \Delta : \mathcal{O} \times \mathcal{O} \longrightarrow \mathbb{S}i\mathbb{S}ets \times \mathbb{S}i\mathbb{S}ets$ and natural transformations φ as in the theorem are in one-to-one correspondence with their restrictions to $\mathcal{O} \times \mathcal{O}$ (via the functor $\Delta \times \Delta$), i.e., natural simplicial maps

$$\varphi_{m,n} : \Delta[m] *_w \Delta[n] \longrightarrow \Delta[m] * \Delta[n] = \Delta[m+n+1], \qquad m,n = -1,0,1,\ldots .$$

Assume that such a family exists. For any $k = -1,0,1,\ldots$ and any operator $\tau : [k] \to [1]$ let

$$\lambda_\tau = \varphi_{k,k}(< \mathrm{id}_{[k]}, \mathrm{id}_{[k]}, \tau >) \;:\; [k] \to [2k+1] \,.$$

Because of the naturality we have

$$\varphi_{m,n}(< \xi, \eta, \tau >) = (\xi * \eta)\lambda_\tau, \qquad \text{for } \xi \in \Delta[m]_k, \eta \in \Delta[n]_k. \tag{3}$$

The fact that $\varphi_{k,k}$ is simplicial implies the relation

$$\lambda_\tau \alpha = (\alpha * \alpha)\lambda_{\tau\alpha} \,, \tag{4}$$

for all operators $\alpha : [k'] \to [k]$. In particular,

$$\lambda_\tau(i) = (\varepsilon_i * \varepsilon_i)\lambda_{\tau\lambda_i}(0)$$

with the vertex operators $\varepsilon_i : [0] \to [k]$ sending 0 to i, for $i = 0,1,\ldots,k$. It follows that the family $(\lambda_\tau)_\tau$ is uniquely determined by the two elements

$$\lambda_{\delta_i} : [0] \to [1],$$

where $\delta_i : [0] \to [1]$ are the usual face operators, for $i = 0, 1$. If the number 1 is not in the image of $\tau : [k] \to [1]$ the element $\varphi_{m,n}(< \xi, \eta, \tau >)$ has to be independent of η. This implies $\lambda_{\delta_1}(0) = 0$, $\lambda_{\delta_0}(0) = 1$ and hence

$$\lambda_\tau(i) = i + (k+1)\tau(i), \quad \text{for } i = 0, 1, \ldots, k\,. \tag{5}$$

Thus the family $(\lambda_\tau)_\tau$ is unique. It follows from (3) that $\varphi_{m,n}$ is unique for all $m, n \geq 0$. The family $(\varphi_{m,-1})_m$ is a natural transformation of the functor $\Delta : \mathcal{O} \to \mathbb{S}i\mathbb{S}ets$ into itself. Using the vertex maps as above it is clear that it has to be the identity.

To show the existence we define the operator λ_τ by (5). Then (4) holds and, working backwards, we obtain first the family $(\varphi_{m,n})_{m,n}$ and then the whole natural transformation φ. Note that it sends $< x, y, \tau > \in \mathbb{X} *_w \mathbb{Y}$ into the class of $(x, y, \tau) \in \mathbb{X} * \mathbb{Y}$ as a quotient of (1).

(b) Define maps

$$\Psi_{m,n} : \Delta^m \times \Delta^n \times I \longrightarrow \Delta^{m+n+1}$$

by

$$\Psi_{m,n}((r_0, \ldots, r_m), (s_0, \ldots, s_n), t) = ((1-t)r_0, \ldots, (1-t)r_m, ts_0, \ldots, ts_n)\,,$$

for $((r_0, \ldots, r_m), (s_0, \ldots, s_n)) \in \Delta^m \times \Delta^n$ and $t \in I$. They induce homeomorphisms

$$\Phi_{m,n} : \Delta^m * \Delta^n \longrightarrow \Delta^{m+n+1}$$

natural in m, n, and extend uniquely to a natural transformation

$$\Phi : |\mathbb{X}| *_k |\mathbb{Y}| \cong |\mathbb{X} *_w \mathbb{Y}| \longrightarrow |\mathbb{X} * \mathbb{Y}|$$

consisting of homeomorphisms because both functors involved are left Kan extensions of their restrictions to $\mathcal{O} \times \mathcal{O}$ (via the functor $\Delta \times \Delta$).

Both maps $|\varphi_{m,n}|$ and $\Phi_{m,n}$ go from $\Delta^m * \Delta^n$ to Δ^{m+n+1}; connect them by linear homotopies $H_{m,n}$. Since the maps which they connect are natural and pairs of operators $[m'] \to [m]$ and $[n'] \to [n]$ induce linear maps $\Delta^{m'} * \Delta^{n'} \to \Delta^m * \Delta^n$, $\Delta^{m'+n'+1} \to \Delta^{m+n+1}$, these homotopies form a family which is natural in m, n. Hence, there is a unique extension of them to a natural homotopy between $|\varphi|$ and Φ. □

Corollary 3.4. *The simplicial map $\varphi : \mathbb{X} *_w \mathbb{Y} \to \mathbb{X} * \mathbb{Y}$ is a weak homotopy equivalence for all simplicial sets $\mathbb{X}$ and $\mathbb{Y}$.*

□

Recall that a simplicial map is called a *weak homotopy equivalence* (WHE) if its geometric realiziation is a honest homotopy equivalence in the category of topological spaces.

Remark 3.5. There does not exist any natural transformation from the join to the weak join of simplicial sets.

If such a transformation existed one could show like in the proof of part (b) of Theorem 3.3 that all its maps are WHE's. But for many simplicial sets $\mathbb{X}, \mathbb{Y}$ there is no WHE

$$\psi : \mathbb{X} * \mathbb{Y} \longrightarrow \mathbb{X} *_w \mathbb{Y}.$$

Take e.g., $\mathbb{X} = \mathbb{Y} = \mathbb{S}^1 = \Delta[1]/\dot{\Delta}[1]$. The simplicial set $\mathbb{S}^1 * \mathbb{S}^1$ is generated by one 3-simplex x with $x\delta_0 = x\delta_1$ and $x\delta_2 = x\delta_3$. On the other hand, $\mathbb{S}^1 *_w \mathbb{S}^1$ is generated by six 3-simplices and each of them has at least one non-degenerated face which is not identified with another one. So, a simplicial map ψ has to assign a degenerated simplex to the generating simplex x. This would lead to the zero homomorphism for the integer homology groups on the level 3 which is no isomorphism because the corresponding homology groups – both simplicial sets involved having the homology of the 3-sphere – do not vanish.

Briefly, we study another join-like construction used in [6]. It arises if the double mapping cylinder is replaced by a homotopy colimit closely related to it. For any functor F from a small category $\mathbb{I}$ into the category of topological spaces, k-spaces or simplicial sets, the homotopy colimit $\operatorname{hocolim} F$ is defined in [3]. In the special case, where $\mathbb{I}$ has three objects and two non-identity morphisms and is described by the diagram

$$i_1 \overset{\alpha}{\longleftarrow} i_0 \overset{\beta}{\longrightarrow} i_2 \tag{6}$$

$\operatorname{hocolim} F$ is the (ordinary) colimit of the diagram

$$F(i_1) \overset{F(\alpha)}{\leftarrow} F(i_0) \overset{j_1}{\rightarrow} F(i_0) \times I \overset{j_0}{\leftarrow} F(i_0) \overset{j_0}{\rightarrow} F(i_0) \times I \overset{j_1}{\leftarrow} F(i_0) \overset{F(\beta)}{\rightarrow} F(i_2), \tag{7}$$

where I is the unit interval or the simplicial set $\Delta[1]$ respectively, and j_0, j_1 are the canonical inclusions into the bottom and the top of the cylinder. Loosely speaking it is the double mapping cylinder $Z(F(\alpha), F(\beta))$ i.e., the colimit of

$$F(i_1) \overset{F(\alpha)}{\longleftarrow} F(i_0) \overset{F(\beta)}{\longrightarrow} F(i_2)$$

with two cylinders inserted instead of one. For topological spaces or k-spaces this makes no difference up to homeomorphism. For simplicial sets there is an obvious map

$$\omega : \operatorname{hocolim} F \longrightarrow Z(F(\alpha), F(\beta))$$

which collapses the first inserted cylinder $F(i_0) \times \Delta[1]$ to $F(i_0)$. Its geometric realization is homotopic to the canonical homeomorphism

$$\operatorname{hocolim} |F| \xrightarrow{\cong} Z(|F(\alpha)|, |F(\beta)|).$$

Hence ω is a WHE.

Specializing to the case where $F(\alpha), F(\beta)$ are the projections of a product $\mathbb{X} \times \mathbb{Y}$ of simplicial sets to its factors we get

Proposition 3.6. *The canonical simplicial map* $\omega : \operatorname{hocolim} F \to \mathbb{X} *_w \mathbb{Y}$ *is a weak homotopy equivalence, for all simplicial sets* $\mathbb{X}$ *and* $\mathbb{Y}$*.*

□

4. Categorical joins.

The join of two small categories $\mathbb{C}$ and $\mathbb{C}'$ can be defined in a complete analogy to the topological case and to the weak join in the simplicial case as the double mapping cylinder of the projection functors

$$\mathbb{C} \longleftarrow \mathbb{C} \times \mathbb{C}' \longrightarrow \mathbb{C}'. \tag{1}$$

In the category $\mathbb{C}at$ of small categories the double mapping cylinder $Z(f_1, f_2)$ of a pair of functors (morphisms in $\mathbb{C}at$)

$$\mathbb{C}_1 \xleftarrow{f_1} \mathbb{C}_0 \xrightarrow{f_2} \mathbb{C}_2 \tag{2}$$

is the colimit of the diagram

$$\mathbb{C}_1 \xleftarrow{f_1} \mathbb{C}_0 \xrightarrow{j_0} \mathbb{C}_0 \times [1] \xleftarrow{j_1} \mathbb{C}_0 \xrightarrow{f_2} \mathbb{C}_2,$$

where j_0 and j_1 are the canonical inclusions into the bottom and the top of the cylinder $\mathbb{C}_0 \times [1]$. Remember that $\mathcal{O}$ is a subcategory of $\mathbb{C}at$.

In general the explicit description of colimits in $\mathbb{C}at$, in particular double mapping cylinders, may be complicated, but the structure of the *join* $\mathbb{C} * \mathbb{C}'$ of small categories $\mathbb{C}$ and $\mathbb{C}'$ is simple. The set of objects in $\mathbb{C} * \mathbb{C}'$ is the disjoint union of the sets of objects in $\mathbb{C}$ and $\mathbb{C}'$. If A and B are objects in $\mathbb{C} * \mathbb{C}'$ then the set $(\mathbb{C} * \mathbb{C}')(A, B)$ of morphisms from A to B in $\mathbb{C} * \mathbb{C}'$ is given as

$$(\mathbb{C} * \mathbb{C}')(A, B) = \begin{cases} \mathbb{C}(A, B), & \text{if } A \text{ and } B \text{ are in } \mathbb{C}; \\ \mathbb{C}'(A, B), & \text{if } A \text{ and } B \text{ are in } \mathbb{C}'; \\ \text{singleton } \{(A, B)\}, & \text{if } A \text{ is in } \mathbb{C} \text{ and } B \text{ is in } \mathbb{C}'; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Thus our join of categories coincides with the one in [13] and extends the join of finite ordinals used in Section 3.

The nerve functor N is a full embedding of $\mathbb{C}at$ into the category $\mathbb{S}i\mathbb{S}ets$ of simplicial sets.

Proposition 4.1. *For small categories $\mathbb{C}$ and $\mathbb{C}'$, there is a natural isomorphism*

$$N\,\mathbb{C} * N\,\mathbb{C}' \xrightarrow{\cong} N(\mathbb{C} * \mathbb{C}')$$

of simplicial sets.

Proof. The natural transformation comes from universal properties of Kan extensions and the fact that the functor $N \times N$ is a full embedding. Explicitly it sends

$$(\beta, \gamma) \in N_m\mathbb{C} \times N_{k-m-1}\mathbb{C}' \subset (N\mathbb{C} * N\mathbb{C}')_k$$

(cf. (2) in Section 3) to $\beta * \gamma \in N_k(\mathbb{C} * \mathbb{C}')$, which shows that it is a bijection (generalizing Lemma 3.1). □

If one identifies $\mathbb{C}at$ with its image in $\mathbb{S}i\mathbb{S}ets$ under N one can interpret this proposition as saying that the category $\mathbb{C}at$ is closed under the join in $\mathbb{S}i\mathbb{S}ets$. On the other hand it is not closed under the weak join, which can be seen by forming $\Delta[0] *_w \Delta[1]$. Then one gets a nondegenerate 2-simplex whose 2nd face is degenerate which cannot happen in the nerve of a small category. In view of Theorem 3.3 one could say that $\mathbb{C}at$ is closed under the weak join in $\mathbb{S}i\mathbb{S}ets$ *up to homotopy*. But this is trivial since every simplicial set has the weak homotopy type of a small category [10].

For all pairs (f_1, f_2) as in (2) there is a natural transformation ζ from the double mapping cylinder $Z(Nf_1, Nf_2)$ in $\mathbb{S}i\mathbb{S}sets$ to $NZ(f_1, f_2)$ coming from the properties of colimits and the fact that N commutes with products. In the special case of (1) this is nothing else than the composition of the WHE

$$\varphi : N\mathbb{C} *_w N\mathbb{C}' \longrightarrow N\mathbb{C} * N\mathbb{C}'$$

of Theorem 3.3 with the isomorphism of Proposition 4.1. This raises the question whether ζ is always a WHE. The answer is no, as shown by the following class of examples. For any small category $\mathbb{C}$ consider

$$[0] * \mathbb{C} \xleftarrow{j} \mathbb{C} \xrightarrow{q} [0],$$

where j is the canonical inclusion (and q unique). Then the geometric realization of $Z(Nj, Nq)$ is homeomorphic to the suspension of the classifying space $B\mathbb{C}$. On the other hand, if $\mathbb{C}$ is connected then the realization of $NZ(j,q)$ is homeomorphic to the join of an interval with $B\mathbb{C}$, hence contractible.

The nerve functor N has a left adjoint $c : \mathbb{S}i\mathbb{S}ets \to \mathbb{C}at$, called *categorization* or *categorical realization* [10]. It commutes with colimits (like any left adjoint) and with finite products (in analogy to geometric realization). Therefore, for any pair of maps

$$\mathbb{X}_1 \xleftarrow{f_1} \mathbb{X}_0 \xrightarrow{f_2} \mathbb{X}_2$$

in $\mathbb{S}i\mathbb{S}sets$ we have a natural isomorphism $cZ(f_1, f_2) \cong Z(cf_1, cf_2)$. As a special case, we get $c(\mathbb{X} *_w \mathbb{Y}) \cong c(\mathbb{X}) * c(\mathbb{Y})$ for all simplicial sets $\mathbb{X}$ and $\mathbb{Y}$. Adjointness properties together with Proposition 4.1 give a natural transformation

$$c(\mathbb{X} * \mathbb{Y}) \longrightarrow c(\mathbb{X}) * c(\mathbb{Y})$$

which turns out to be an isomorphism. To check the latter it is enough to consider the case $\mathbb{X} = \Delta[m]$, $\mathbb{Y} = \Delta[n]$ and to use compatibility with colimits over connected diagrams. In the same way it is easy to check that c transforms the simplicial map $\varphi : \mathbb{X} *_w \mathbb{Y} \to \mathbb{X} * \mathbb{Y}$ into the composition of the two isomorphisms just described.

Finally, we look at constructions in $\mathbb{C}at$ which are related to the homotopy colimits mentioned at the end of Section 3. Given a covariant functor F from a small category $\mathbb{I}$ into the category $\mathbb{C}at$ the Grothendieck construction $\mathbb{I} \int F$ (see e.g., [23]) is defined. The objects in the category $\mathbb{I} \int F$ are the pairs (i, X) with i an object in $\mathbb{I}$ and X an object in $F(i)$, the morphisms from (i, X) to (i', X') are the pairs (α, x) consisting of a morphism $\alpha : i \to i'$ in $\mathbb{I}$ and a morphism $x : F(\alpha)(X) \to X'$ in $F(i')$; composition is defined in the canonical way.

In particular, if $\mathbb{I}$ is the category represented by the diagram

$$i_1 \xleftarrow{\alpha} i_0 \xrightarrow{\beta} i_2 \tag{3}$$

as in (6) of Section 3 then $\mathbb{I} \int F$ is the colimit in $\mathbb{C}at$ of the diagram

$$F(i_1) \overset{F(\alpha)}{\leftarrow} F(i_0) \overset{j_1}{\to} F(i_0) \times [1] \overset{j_0}{\leftarrow} F(i_0) \overset{j_0}{\to} F(i_0) \times [1] \overset{j_1}{\leftarrow} F(i_0) \overset{F(\beta)}{\to} F(i_2), \tag{4}$$

hence completely analogous to the homotopy colimit (7) in Section 3. Like in that case, we have a canonical map (functor) π from $\mathbb{I}\int F$ to the double mapping cylinder $Z(F(\alpha), F(\beta))$ which collapses the first cylinder $F(i_0) \times [1]$ in (4) to $F(i_0)$. Unlike in the case of simplicial sets the functor π is not always a WHE. (A functor between small categories is called a WHE if N transforms it into a simplicial WHE). However, if we specialize further to the situation where $F(\alpha), F(\beta)$ are the projections of a product of two small categories $\mathbb{C}$ and $\mathbb{C}'$ then we do have

Proposition 4.2. *The canonical functor π from $\mathbb{I}\int F$ to $Z(F(\alpha), F(\beta)) = \mathbb{C} * \mathbb{C}'$ is a weak homotopy equivalence for all small categories $\mathbb{C}$ and $\mathbb{C}'$.*

Hence, in this special case, the Grothendieck construction can be considered as a variant of the categorical join.

In order to prove Proposition 4.2 and to understand the whole situation recall from [23] that for any small category $\mathbb{I}$ and any functor $F : \mathbb{I} \to \mathbb{C}at$, the Grothendieck construction $\mathbb{I}\int F$ is canonically isomorphic to $c(\operatorname{hocolim} NF)$ and the adjunction unit $\eta : \mathrm{Id} \to Nc$ gives a WHE $\operatorname{hocolim} NF \longrightarrow N(\mathbb{I}\int F)$. If $\mathbb{I}$ is as in (3) we obtain the commutative diagram

$$\begin{array}{ccc} \operatorname{hocolim} NF & \xrightarrow{\ \omega\ } & Z(NF(\alpha), NF(\beta)) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\zeta} \\ N(\mathbb{I}\int F) & \xrightarrow{\ N\pi\ } & NZ(F(\alpha), F(\beta)). \end{array}$$

Since η and ω are both WHE's, $N\pi$ is a WHE if and only if ζ is. Above we saw that ζ is not always a WHE, but it is in the case of the join. Thus the same holds for π.

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Anschrift der Autoren:

Rudolf Fritsch
Mathematisches Institut
Ludwig-Maximilians-Universität München
Theresienstraße 39
D-80333 München
Germany
e-mail: fritsch@mathematik.uni-muenchen.de

Marek Golansiński
Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń
Poland
e-mail: marek@mat.uni.torun.pl