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An n–dimensional Bodenmiller Theorem

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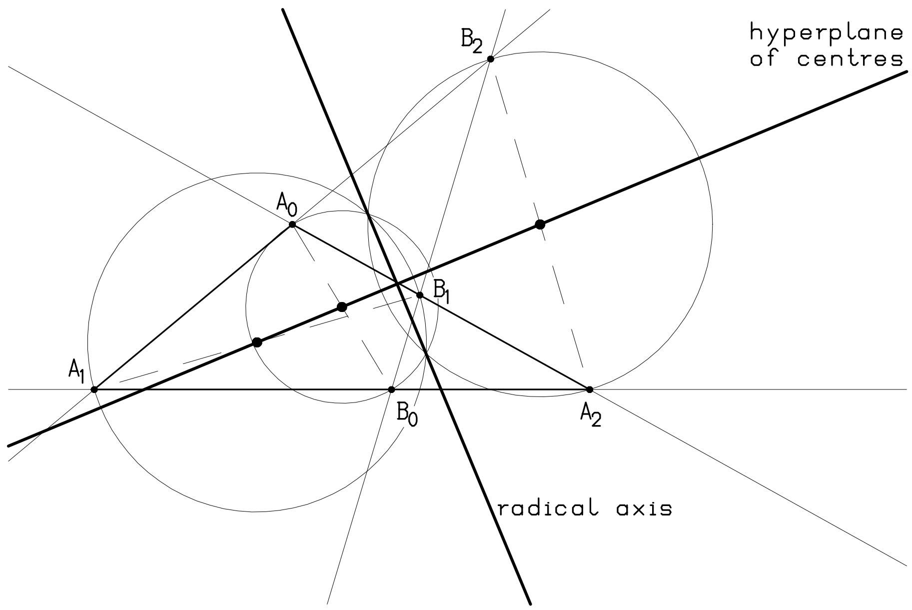

Bodenmiller's Theorem using the notation of Theorems 1 and 2

The following is an answer to Chris Fisher's query in [2]: Is there an n-dimensional version of Bodenmiller's Theorem? Yes, there is one, but not as Fisher conjectured. In order to make this note self-contained we repeat some facts from [2], [3], [4].

The original version of BODENMILLER's Theorem states that the three circles with the diagonals of a complete quadrilateral as diameters intersect in the same two points; more precisely, they belong to a pencil of coaxal circles [1, p. 35]. As a consequence the midpoints of these diagonals are collinear, a fact already known to Newton and Gauß. The connection to Bodenmiller's Theorem has been exhibited by Schlömilch and Möbius; for precise references to its history see [3]. In some textbooks on geometry, e.g. [7, p. 159] it may be found in the following form:

Gauss' Theorem. *If the sides (suitably extended) of triangle $A_0A_1A_2$ are cut by a straight line in three distinct points $B_0 \in A_1A_2$, $B_1 \in A_2A_0$, $B_2 \in A_0A_1$ then the midpoints of the line segments $[A_i, B_i]$, $i \in \{0, 1, 2\}$, are collinear.* □

This theorem has been generalized to n-space as a problem posed by Murray S. Klamkin:

Theorem 1. *Let $\sigma = A_0A_1A_2 \dots A_n$ be an n-simplex and let $\mathcal{A}_i$ denote the hyperplane containing the $(n-1)$-dimensional face of σ opposite the vertex A_i, $i \in \{0, 1, \dots, n\}$. If these hyperplanes are cut by a straight line in $n+1$ distinct points $B_i \in \mathcal{A}_i$, $i \in \{0, 1, \dots, n\}$, then the midpoints of the line segments $[A_i, B_i]$, $i \in \{0, 1, \dots, n\}$, lie in the same hyperplane.*

Ivan Paasche's solution [6] was translated into English for *Crux* [2]. In a comment appended to that solution, Fisher asked if this theorem can be obtained in general – as in the case $n = 2$ – from a *"Bodenmiller Theorem"*. More precisely, do the hyperspheres having the segments $[A_i, B_i]$ as diameters intersect in the same $(n-2)$-sphere? The answer is *no*, but there is positive answer in terms of a collection of hyperspheres that we shall call a *system of coaxal spheres*, cf. [5, p. 34]:

Theorem 2. *Under the assumptions of Theorem 1, the spheres $\mathcal{S}_i$ having the line segments $[A_i, B_i]$ as diameters, $i \in \{0, 1, \ldots, n\}$, belong to a system of coaxal spheres.* Theorem 2 implies Theorem 1, as will be shown later on. One geometric meaning of Theorem 2 is the following: If n of the $n+1$ spheres $\mathcal{S}_i$ have two points in common, then the remaining sphere also passes through these two points.

Theorem 2 is true in a very general setting. Let K be a field, V a K-vector space with $\dim_K V = n < \infty$, and $<,>$ a nondegenerate, symmetric bilinear form

$$<,>: V \times V \longrightarrow K\,,\ \ (\boldsymbol{x}, \boldsymbol{y}) \mapsto <\boldsymbol{x}, \boldsymbol{y}> = \boldsymbol{x} \cdot \boldsymbol{y}.$$

We consider a *sphere* to be a formal polynomial

$$P = P(\boldsymbol{x}) = \boldsymbol{x}^2 - \boldsymbol{c} \cdot \boldsymbol{x} + d$$

with $\boldsymbol{c} \in V$ and $d \in K$. For example, in the Euclidean plane a circle with centre $(\frac{c_1}{2}, \frac{c_2}{2})$ and radius r has $\boldsymbol{c} = (c_1, c_2)$, $d = \frac{c_1^2}{4} + \frac{c_2^2}{4} - r^2$: any point $\boldsymbol{x} = (x, y)$ on the circle satisfies $x^2 + y^2 - (c_1 x + c_2 y) + \frac{c_1^2}{4} + \frac{c_2^2}{4} - r^2 = (x - \frac{c_1}{2})^2 + (y - \frac{c_2}{2})^2 - r^2 = 0$. Although this circle is the zero set of the function $P(\boldsymbol{x})$, from our general point of view distinct polynomials having the same zero sets are to be considered as distinct spheres; in particular, this applies to polynomials with empty or one-point zero sets to be admitted.

The 2-set $\{\boldsymbol{a}, \boldsymbol{b}\}$ is called a *diameter* of the sphere P if $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ and $d = \boldsymbol{a} \cdot \boldsymbol{b}$. The first condition places the centre – if it exists (see the final paragraph) – at $\boldsymbol{c}/2$; the second condition forces $\boldsymbol{a}$ and $\boldsymbol{b}$ to satisfy the equation $P(\boldsymbol{x}) = 0$. Thus a sphere with diameter $\{\boldsymbol{a}, \boldsymbol{b}\}$ is the locus of points $\boldsymbol{x}$ such that the triangle $\boldsymbol{axb}$ has a right angle at $\boldsymbol{x}$: $0 = (\boldsymbol{x} - \boldsymbol{a}) \cdot (\boldsymbol{x} - \boldsymbol{b}) = \boldsymbol{x}^2 - (\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{x} + \boldsymbol{a} \cdot \boldsymbol{b}$.

A set $\mathcal{P}$ of spheres is called an *(ordinary) system of coaxal spheres,* or *coaxal system* for short, if there are spheres $P_i = \boldsymbol{x}^2 - \boldsymbol{c}_i \cdot \boldsymbol{x} + d_i \in \mathcal{P}$, $i \in \{1, \ldots, n\}$ such that the family of vectors $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n)$ is affinely independent[1] and

$$\mathcal{P} = \{P = \boldsymbol{x}^2 - (\sum_{i=1}^{n} t_i \boldsymbol{c}_i) \cdot \boldsymbol{x} + \sum_{i=1}^{n} t_i d_i \mid t_1, \ldots, t_n \in K, \sum_{i=1}^{n} t_i = 1\}\,.$$

(When $n = 2$ the family is usually called a *coaxal pencil of circles*; when $n = 3$ it is a *coaxal bundle of spheres.*) A family $(P_1, \ldots, P_n)$ of spheres that define the system $\mathcal{P}$ is called a *set of generators* for $\mathcal{P}$; the solution set of the system of equations

$$(\boldsymbol{c}_i - \boldsymbol{c}_1) \cdot \boldsymbol{x} = d_i - d_1\,,\ \ i \in \{2, \ldots, n\}$$

is a straight line, called the *(radical) axis* of $\mathcal{P}$. The axis of a system is independent of the choice of generators.

Now we turn to a proof of Theorem 2. Let σ be the given n-simplex – an affinely independent family $\sigma = (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in V^{n+1}$. For $i \in \{0, 1, \ldots, n\}$ the *face* $\mathcal{A}_i$ is the hyperplane containing the vertices $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\ldots$, $\boldsymbol{a}_{i-1}$, $\boldsymbol{a}_{i+1}$, $\ldots$, $\boldsymbol{a}_n$. Without loss of generality we assume $\boldsymbol{a}_0 = 0$, implying the family $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ to be a basis of V. Also, let

$$\mathcal{L} = \{\vec{\boldsymbol{b}} + s\boldsymbol{b} \mid s \in K\}$$

be a straight line cutting the faces $\mathcal{A}_i$ in $n+1$ distinct points $\boldsymbol{b}_i$, $i \in \{0, 1, \ldots, n\}$. Also without loss of generality, we assume $\vec{\boldsymbol{b}}$, $\boldsymbol{b} \in \mathcal{A}_0$, so that

$$\vec{\boldsymbol{b}} = \boldsymbol{b}_0 = \sum_{i=1}^{n} r_i \boldsymbol{a}_i\,,\ \ \boldsymbol{b} = \sum_{i=1}^{n} t_i \boldsymbol{a}_i\,,$$

[1] This is the case if and only if the vectors of the family $(c_2 - c_1, \ldots, c_n - c_1)$ are linearly independent.

with

$$\sum_{i=1}^{n} r_i \;=\; \sum_{i=1}^{n} t_i \;=\; 1\,.$$

Then we have

$$\boldsymbol{b}_i = \boldsymbol{b}_0 + s_i\boldsymbol{b}$$

with

$$r_i + s_i \cdot t_i = 0$$

for all $i \in \{1,\dots,n\}$, all r_i, s_i, t_i being different from zero.

Next, we have to consider the spheres $P_0 = \boldsymbol{x}^2 - \boldsymbol{b}_0 \cdot \boldsymbol{x}$ and $P_i = \boldsymbol{x}^2 - (\boldsymbol{a}_i + \boldsymbol{b}_i)\cdot\boldsymbol{x} + \boldsymbol{a}_i \cdot \boldsymbol{b}_i$, $i \in \{1,\dots,n\}$; we claim that the family $(P_1,\dots,P_n)$ generates a coaxal system containing P_0.

To prove this, compute

$$(\boldsymbol{a}_i+\boldsymbol{b}_i)-(\boldsymbol{a}_1+\boldsymbol{b}_1) = \boldsymbol{a}_i-\boldsymbol{a}_1+(s_i - s_1)\boldsymbol{b}$$

for $i \in \{2,\dots,n\}$. In order to show the required linear independence consider the equation

$$\sum_{i=2}^{n} u_i(\boldsymbol{a}_i-\boldsymbol{a}_1+(s_i - s_1)\boldsymbol{b}) = 0$$

with $u_i \in K$, not all u_i equal zero. As a consequence of the linear independence of the family $(\boldsymbol{a}_2-\boldsymbol{a}_1,\dots,\boldsymbol{a}_n-\boldsymbol{a}_1)$,

$$\sum_{i=2}^{n} u_i \cdot (s_i - s_1) \neq 0\,.$$

Thus, we may assume

$$\sum_{i=2}^{n} u_i \cdot (s_i - s_1) = -1\,,$$

so that

$$\boldsymbol{b} = \sum_{i=2}^{n} u_i(\boldsymbol{a}_i-\boldsymbol{a}_1) = \sum_{i=2}^{n} u_i\boldsymbol{a}_i - (\sum_{i=2}^{n} u_i)\boldsymbol{a}_1\,.$$

This implies

$$t_1 = -\sum_{i=2}^{n} u_i\,, \qquad t_i = u_i\,, \quad i \in \{2,\dots,n\}\,,$$

yielding the contradiction

$$\sum_{i=1}^{n} t_i = 0\,.$$

It remains to be shown that P_0 belongs to the coaxal system generated by the family $(P_1,\dots,P_n)$. As a matter of fact,

$$\begin{aligned}\sum_{i=1}^{n} t_i(\boldsymbol{a}_i+\boldsymbol{b}_0+s_i\boldsymbol{b}) \;&=\; \sum_{i=1}^{n} t_i\boldsymbol{a}_i+(\sum_{i=1}^{n} t_i)\boldsymbol{b}_0+(\sum_{i=1}^{n} t_i s_i)\boldsymbol{b} \;=\\ &=\; \boldsymbol{b} + \boldsymbol{b}_o-(\sum_{i=1}^{n} r_i)\boldsymbol{b} \;=\; \boldsymbol{b}_0,\end{aligned}$$

and

$$\sum_{i=1}^{n} t_i \boldsymbol{a}_i \cdot (\boldsymbol{b}_0 + s_i \boldsymbol{b}) = (\sum_{i=1}^{n} t_i \boldsymbol{a}_i) \cdot \boldsymbol{b}_0 + (\sum_{i=1}^{n} t_i s_i \boldsymbol{a}_i) \cdot \boldsymbol{b} =$$

$$= \boldsymbol{b} \cdot \boldsymbol{b}_0 - (\sum_{i=1}^{n} r_i \boldsymbol{a}_i) \cdot \boldsymbol{b} = 0,$$

which proves Theorem 2. □

In order to derive Theorem 1 from Theorem 2 we need a further assumption. The characteristic of the underlying field K must be different from 2, because otherwise we would not be able to find midpoints of line segments and centers of spheres. Under this assumption the center of the sphere $P = \boldsymbol{x}^2 - \boldsymbol{c} \cdot \boldsymbol{x} + d$ is the vector $1/2 \cdot \boldsymbol{c}$. Given an n-simplex $\sigma = (\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in V^{n+1}$ and a straight line $\mathcal{L}$ cutting the faces $\mathcal{A}_i$ in distinct points $\boldsymbol{b}_i$, Theorem 2 establishes that the family $(\boldsymbol{a}_0 + \boldsymbol{b}_0, \boldsymbol{a}_1 + \boldsymbol{b}_1, \ldots, \boldsymbol{a}_n + \boldsymbol{b}_n)$ is affinely dependent (i.e., belongs to a hyperplane). The dilatation centered at the origin with ratio 1/2 transforms hyperplanes into hyperplanes; thus, the centers of the spheres under consideration – agreeing with the midpoints of the line segments in the classical situation – belong to a hyperplane. □

References

[1] Coxeter, Harold Scott MacDonald and Greitzer, Samuel L., *Geometry Revisited,* New Mathematical Library, volume 19, Random House, The L.W. Singer Company, New York – Toronto 1967.

[2] Fisher, Chris, Editor's comment to problem 1898, *Crux Mathematicorum,* **20** (1994), pp. 265-266.

[3] Fritsch, Rudolf, Gudermann, Bodenmiller und der Satz von Bodenmiller - Steiner, *Didaktik der Mathematik* **20** (1992) 165–187.

[4] Fritsch, Rudolf, Remarks on Bodenmiller's Theorem, *Journal of Geometry* **47** (1993) 23–31

[5] Johnson, Roger Arthur, *Advanced Euclidean Geometry: An Elementary treatise on ther Geometry of the Triangle and the Circle (Modern geometry).* Dover Publications, Inc., New York. 1960.

[6] Paasche, Ivan, Lösung von Aufgabe 733, *Elemente der Mathematik* **31** (1976) 15.

[7] Скопец, Залман Алтерович, *Геометрические Миниатюры*, Просвещение, Moscow, 1990.

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