Mathematisches Institut
der Universität München
Prof. Otto Forster

SoSe 2024
May 8, 2024

Selected Topics from Number Theory

Problem sheet #3

Problem 9 The sequence $(f_\nu)_{\nu\geqslant 0}$ of Fibonacci numbers is recursively defined by

$$f_0 := 0,\ \ f_1 := 1, \qquad f_{n+1} := f_n + f_{n-1}, \quad (n \geqslant 1).$$

a) Show that the limit $\lim\limits_{n\to\infty} \dfrac{f_{n+1}}{f_n}$ exists and equals the *golden ration*

$$\phi := \frac{1+\sqrt{5}}{2}.$$

Problem 10 (Continuation of Problem 9)
a) Show that the CF expansion of ϕ is

$$\phi = \mathrm{cfrac}(1,1,1,1,1,\ldots)$$

and that the n-th convergent $\dfrac{p_n}{q_n}$ of this continued fraction equals $\dfrac{f_{n+2}}{f_{n+1}}$.
b) Prove that for every constant $c > \sqrt{5}$ the inequality

$$\left|\phi - \frac{p}{q}\right| < \frac{1}{cq^2}, \quad p, q \text{ positive integers,}$$

has only a finite number of solutions.

Problem 11 Sylvester's sequence $(S_n)_{n\geqslant 0}$ is recursively defined by

$$S_0 := 2, \qquad S_n := 1 + \prod_{\nu=0}^{n-1} S_\nu, \quad (n \geqslant 1).$$

Hence the series begins with $(2, 3, 7, 43, 1807, 3263443, \ldots)$.
a) Show that the series can also be defined by

$$S_0 := 2, \qquad S_{n+1} := S_n^2 - S_n + 1, \quad (n \geqslant 0).$$

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p.t.o.

b) Prove that

$$\sum_{n=0}^{\infty} \frac{1}{S_n} = 1.$$

Hint. Show that for every $m \geqslant 1$ one has

$$1 = \sum_{nu=0}^{m-1} \frac{1}{S_\nu} + \frac{1}{S_m - 1}.$$

Problem 12 (Continuation of Problem 11)
Cahen's constant is defined by

$$C := \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{S_\nu - 1}$$

a) Show that another way to define C is

$$C := \sum_{\nu=0}^{\infty} \frac{1}{S_{2\nu}}.$$

b) Consider the CF expansion $C = \mathrm{cfrac}(a_0, a_1, a_2, a_3, \ldots)$ and prove that all coefficients a_ν are squares, in fact

$$C = \mathrm{cfrac}(0, 1, 1, 1, 4, 9, 196, 16641, \ldots).$$

These problems will be discussed Wednesday, May 22, 2024, 16-18 h