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Selected Topics from Number Theory

Problem sheet #1, Solutions

Problem 1 Calculate the CF (= continued fraction) expansion of the numbers

$$x_1 := \sqrt{7}, \qquad x_2 := \frac{\sqrt{7}}{2}, \qquad x_3 := \frac{\sqrt{7}}{3}.$$

Solution

a) $\sqrt{7} = \text{cfrac}(2, \overline{1, 1, 1, 4}),$

b) $\dfrac{\sqrt{7}}{2} = \text{cfrac}(1, \overline{3, 10, 3, 2}),$

c) $\dfrac{\sqrt{7}}{3} = \text{cfrac}(0, 1, \overline{7, 2})$

As an example, we give a proof of b)

$$\frac{\sqrt{7}}{2} = 1 + \frac{\sqrt{7}-2}{2} = 1 + \frac{1}{\xi_1},$$

$$\xi_1 = \frac{2}{\sqrt{7}-2} = \frac{2(\sqrt{7}+2)}{3} = 3 + \frac{2\sqrt{7}-5}{3} = 3 + \frac{1}{\xi_2},$$

$$\xi_2 = \frac{3}{2\sqrt{7}-5} = \frac{6\sqrt{7}+15}{3} = 2\sqrt{7}+5 = 10 + (2\sqrt{7}-5) = 10 + \frac{1}{\xi_3},$$

$$\xi_3 = \frac{1}{2\sqrt{7}-5} = \frac{2\sqrt{7}+5}{3} = 3 + \frac{2\sqrt{7}-4}{3} = 3 + \frac{1}{\xi_4},$$

$$\xi_4 = \frac{3}{2\sqrt{7}-4} = \frac{3(2\sqrt{7}+4)}{12} = \frac{\sqrt{7}+2}{2} = 2 + \frac{\sqrt{7}-2}{2} = 2 + \frac{1}{\xi_5},$$

$$\xi_5 = \frac{2}{\sqrt{7}-2} = \xi_1.$$

Problem 2 Let k be a natural number. Derive formulas for the CF expansions of the numbers

a) $\sqrt{k^2+1}, \quad (k \geqslant 1)$

b) $\sqrt{k^2-1}, \quad (k \geqslant 2),$

c) $\sqrt{k^2+2}, \quad (k \geqslant 1),$

d) $\sqrt{k^2-2}, \quad (k \geqslant 3).$

Solution

a) $\sqrt{k^2+1} = \mathrm{cfrac}(k, \overline{2k}), \quad k \geqslant 1,$

b) $\sqrt{k^2-1} = \mathrm{cfrac}(k{-}1, \overline{1, 2k{-}2}), \quad k \geqslant 2,$

c) $\sqrt{k^2+2} = \mathrm{cfrac}(k, \overline{k, 2k}), \quad k \geqslant 1,$

d) $\sqrt{k^2-2} = \mathrm{cfrac}(k{-}1, \overline{1, k{-}2, 1, 2k{-}2}), \quad k \geqslant 3.$

Proof of d) The following identities will be used:

$$\begin{aligned}(k^2-2)-(k-1)^2 &= k^2-2-(k^2-2k+1) = 2k-3,\\ (k^2-2)-(k-2)^2 &= k^2-2-(k^2-4k+4) = 4k-6.\end{aligned}$$

Now we begin the CF expansion of $\sqrt{k^2-2}$

$$\sqrt{k^2-2} = (k-1) + (\sqrt{k^2-2}-(k-1)) = (k-1) + \frac{1}{\xi_1}$$

$$\begin{aligned}
\xi_1 &= \frac{1}{\sqrt{k^2-2}-(k-1)} = \frac{\sqrt{k^2-2}+(k-1)}{2k-3} = 1 + \frac{\sqrt{k^2-2}-(k-2)}{2k-3} = 1 + \frac{1}{\xi_2},\\
\xi_2 &= \frac{2k-3}{\sqrt{k^2-2}-(k-2)} = \frac{(2k-3)(\sqrt{k^2-2}+(k-2)}{4k-6} = \frac{\sqrt{k^2-2}+(k-2)}{2}\\
&= (k-2) + \frac{\sqrt{k^2-2}-(k-2)}{2} = (k-2) + \frac{1}{\xi_3},\\
\xi_3 &= \frac{2}{\sqrt{k^2-2}-(k-2)} = \frac{2(\sqrt{k^2-2}+(k-2))}{4k-6} = \frac{\sqrt{k^2-2}+(k-2)}{2k-3}\\
&= 1 + \frac{\sqrt{k^2-2}-(k-1)}{2k-3} = 1 + \frac{1}{\xi_4},\\
\xi_4 &= \frac{2k-3}{\sqrt{k^2-2}-(k-1)} = \frac{(2k-3)(\sqrt{k^2-2}+(k-1))}{2k-3} = \sqrt{k^2-2}+(k-1)\\
&= (2k-2) + (\sqrt{k^2-2}-(k-1)) = (2k-2) + \frac{1}{\xi_5},\\
\xi_5 &= \frac{1}{\sqrt{k^2-2}-(k-1)} = \xi_1.
\end{aligned}$$

Problem 3 Let k be a natural number. Derive formulas for the CF expansions of the numbers

a) $\sqrt{k^2+4}, \quad (k \geqslant 2),$

b) $\sqrt{k^2-4}, \quad (k \geqslant 5).$

Hint. Distinguish the cases k even and k odd.

Solution

a1) $\sqrt{k^2+4} = \mathrm{cfrac}(k, \overline{(k-1)/2,\ 1,\ 1,\ (k-1)/2,\ 2k}).\quad k$ odd, $k \geqslant 3$.

a2) $\sqrt{k^2+4} = \mathrm{cfrac}(k, \overline{k/2, 2k}),\quad k$ even, $k \geqslant 2$.

b1) $\sqrt{k^2-4} = \mathrm{cfrac}(k-1,\ \overline{1,\ (k-3)/2,\ 2,\ (k-3)/2,\ 1,\ 2k-2}),\quad k$ odd, $k \geqslant 5$,

b2) $\sqrt{k^2-4} = \mathrm{cfrac}(k-1,\ \overline{1,\ k/2-2,\ 1,\ 2k-2})\quad k$ even, $k \geqslant 6$.

Proof of a2)
Since $k \geqslant 2$, one has $k < \sqrt{k^2+4} < k+1$. Remember also that k is even.

$$\sqrt{k^2+4} = k + (\sqrt{k^2+4} - k) = k + \frac{1}{\xi_1},$$
$$\xi_1 = \frac{1}{\sqrt{k^2+4}-k} = \frac{\sqrt{k^2+4}+k}{4} = \frac{k}{2} + \frac{\sqrt{k^2+4}-k}{4} = \frac{k}{2} + \frac{1}{\xi_2},$$
$$\xi_2 = \frac{4}{\sqrt{k^2+4}-k} = \frac{4(\sqrt{k^2+4}+k)}{4} = 2k + (\sqrt{k^2+4}-k) = 2k + \frac{1}{\xi_3},$$
$$\xi_3 = \frac{1}{\sqrt{k^2+4}-k} = \xi_1.$$

Problem 4 Let

$$x = a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \ldots =: \mathrm{cfrac}(a_0, a_1, a_2, a_3, \ldots)$$

be the CF expansion of an irrational real number x. Determine the CF expansion of $-x$.

Solution

i) If $a_1 > 1$, then

$$-x = \mathrm{cfrac}(-a_0-1, 1, a_1-1, a_2, a_3, a_4, \ldots),$$

ii) If $a_1 = 1$, then

$$-x = \mathrm{cfrac}(-a_0-1, a_2+1, a_3, a_4, a_5, \ldots).$$

Proof. It is clear that it suffices to consider only the case $a_0 = 0$.
We begin with case ii).
We define $\xi := \mathrm{cfrac}(a_3, a_4, a_5, ...)$. Then with $a_0 = 0$, $a_1 = 1$ and $b := a_2$ we have

$$x = \mathrm{cfrac}(a_0, a_1, a_2, \xi) = \mathrm{cfrac}(0, 1, b, \xi)$$
$$= \frac{1}{1 + \frac{1}{b + 1/\xi}} = \frac{1}{1 + \frac{\xi}{b\xi + 1}} = \frac{b\xi + 1}{(b+1)\xi + 1}.$$

Define $y := \mathrm{cfrac}(0, b+1, \xi)$. Then

$$y = \frac{1}{b+1+1/\xi} = \frac{\xi}{(b+1)\xi+1}.$$

Therefore

$$x + y = \frac{b\xi+1}{(b+1)\xi+1} + \frac{\xi}{(b+1)\xi+1} = 1,$$

hence $y = 1 - x$. This implies assertion ii).

To prove i), we use the same calculations. Set $b' := b+1 > 1$. Then $y = \mathrm{cfrac}(0, b', \xi)$. Hence

$$1 - y = x = \mathrm{cfrac}(0, 1, b, \xi) = \mathrm{cfrac}(0, 1, b'-1, \xi).$$

Therefore, if $X = \mathrm{cfrac}(0, B, \xi)$ with $B > 1$, then $-X = \mathrm{cfrac}(-1, 1, B-1, \xi)$.
This is assertion i).
