Mathematisches Institut
der Universität München
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Riemann Surfaces

Problem sheet #12

Problem 45

Let $\pi : X \to \mathbb{P}^1$ be the Riemann surface of $\sqrt[n]{1-z^n}$, i.e., of the algebraic function defined by the polynomial

$$w^n + z^n - 1 \in \mathcal{M}(\mathbb{P}^1)[w], \qquad \mathcal{M}(\mathbb{P}^1) \cong \mathbb{C}[z].$$

Prove that the genus of X is

$$g = \frac{(n-1)(n-2)}{2}.$$

Problem 46

Let $P(z) = \prod_{j=1}^4 (z - a_j) \in \mathbb{C}[z]$ be a polynomial of degree 4 with pairwise distinct roots $a_1, \ldots, a_4 \in \mathbb{C}$ and let $\pi : X \to \mathbb{P}^1$ be the Riemann surface of $\sqrt[3]{P(z)}$, i.e., of the algebraic function defined by the polynomial

$$w^3 - P(z) \in \mathcal{M}(\mathbb{P}^1)[w], \qquad \mathcal{M}(\mathbb{P}^1) \cong \mathbb{C}(z).$$

Prove that the genus of X equals 3 and that the differential forms

$$\sigma_1 := \frac{dz}{w}, \quad \sigma_2 := \frac{dz}{w^2}, \quad \sigma_3 := \frac{zdz}{w^2}$$

are holomorphic on X and constitute a basis of $\Omega(X)$.

Problem 47

On a compact Riemann surface X let $\mathcal{Q} \subset \mathcal{M}^{(1)}$ be the sheaf of all meromorphic 1-forms ω on X having residue 0 at every pole of ω. Recall (cf. Problem 26b) that there is a short exact sequence of sheaves $\quad 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{M} \xrightarrow{d} \mathcal{Q} \longrightarrow 0.$

a) Prove that there is an isomomorphism

$$H^1(X, \mathbb{C}) \cong \mathcal{Q}(X)/d\mathcal{M}(X).$$

b) Suppose $X = \mathbb{C}/\Lambda$, with a lattice $\Lambda \subset \mathbb{C}$. Prove that on the torus X the classes of

$$\omega_1 := dz \quad \text{and} \quad \omega_2 := \wp_\Lambda dz$$

form a basis of $\mathcal{Q}(X)/d\mathcal{M}(X) \cong H^1(X, \mathbb{C})$.

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p.t.o.

Problem 48*

Let X be a compact Riemann surface, $a \in X$, and (U, z) a coordinate neighborhood of a such that $z(a) = 0$ and $z(U) = \{z \in \mathbb{C} : |z| < 1\}$.

a) Let D be a divisor on X. Prove that $\mathfrak{U} := (U, X \smallsetminus \{a\})$ is a Leray covering of X for the sheaf $\mathcal{O}_D$, i.e.

$$H^1(X, \mathcal{O}_D) \cong H^1(\mathfrak{U}, \mathcal{O}_D).$$

b) Let g be the genus of X. Denote by $L \subset \mathbb{C}^{2g-1}$ the subspace of all $(2g-1)$-tuples $(c_1, c_2, \ldots, c_{2g-1}) \in \mathbb{C}^{2g-1}$ such that there exists a function $f \in \mathcal{M}(X)$ which is holomorphic in $X \smallsetminus \{a\}$ and whose principal part at a is

$$h(z) = \sum_{\nu=1}^{2g-1} \frac{c_\nu}{z^\nu}.$$

Prove that

$$H^1(X, \mathcal{O}) \cong \mathbb{C}^{2g-1}/L.$$

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*A correct, independently worked out solution of the starred problem is rewarded by a bonus of 0.3 in the final grading.

Please turn in your solution during the lecture or by email to

`forster@math.lmu.de`