Mathematisches Institut
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Riemann Surfaces

Problem sheet #11

Problem 41

Let $F : X \to Y$ be a non-constant holomorphic mapping of compact Riemann surfaces. Let $\omega \in \mathcal{M}^{(1)}(X)$ be a meromorphic differential form on X. Use Problem 40 to define the trace $\sigma := \mathrm{Tr}(\omega) \in \mathcal{M}^{(1)}(Y)$.

a) Show that

$$\sum_{x \in X} \mathrm{Res}_x(\omega) = \sum_{y \in Y} \mathrm{Res}_y(\sigma).$$

b) By representing X as a branched holomorphic covering of the Riemann sphere $\mathbb{P}^1$, prove

$$\sum_{x \in X} \mathrm{Res}_x(\omega) = 0.$$

Problem 42

As in Problem 19, let $p : X_3 \to \mathbb{P}^1$ be the Riemann surface of $\sqrt[3]{1 - z^3}$, i.e. of the algebraic function defined by the polynomial

$$w^3 + z^3 - 1 \in \mathcal{M}(\mathbb{P}^1)[w], \qquad \mathcal{M}(\mathbb{P}^1) \cong \mathbb{C}(z).$$

a) Calculate the divisor of the differential form $\omega := dw \in \mathcal{M}^{(1)}(X_3)$.

b) Calculate the trace $\mathrm{Tr}(\omega) \in \mathcal{M}^{(1)}(\mathbb{P}^1)$.

Problem 43

Let X and Y be compact Riemann surfaces. Suppose X has genus 1 and $F : X \to Y$ is a non-constant holomorphic map.

Prove that either $Y \cong \mathbb{P}^1$, or else Y has genus 1 and F is an unbranched covering map.

Problem 44

a) Let X be a compact Riemann surface of genus $g > 0$. Prove that for every point $a \in X$ there exists a holomorphic 1-form $\omega \in \Omega(X)$ with $\omega(a) \neq 0$.

b) Suppose that the genus of X equals 2. Let (ω_1, ω_2) be a basis of $\Omega(X)$ and define $f \in \mathcal{M}(X)$ by $\omega_1 = f\omega_2$. Show that $f : X \to \mathbb{P}^1$ is a 2-sheeted (branched) covering map.
