Mathematisches Institut
der Universität München
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Riemann Surfaces

Problem sheet #9

Problem 33

Suppose $g \in \mathcal{E}(\mathbb{C})$ is a smooth function with compact support. Prove that there is a solution $f \in \mathcal{E}(\mathbb{C})$ of the equation

$$\frac{\partial f}{\partial \bar{z}} = g$$

having compact support if and only if

$$\iint_{\mathbb{C}} z^n g(z) dz \wedge d\bar{z} = 0 \quad \text{for every } n \geqslant 0.$$

Problem 34

(a) Let X be a manifold, $U \subset X$ open and $V \Subset U$ a relatively compact open subset. Show that V meets only a finite number of connected components of U.

(b) Let X be a compact manifold and $\mathfrak{U} = (U_i)_{i \in I}$, $\mathfrak{V} = (V_i)_{i \in I}$ be two finite open coverings of X such that $V_i \Subset U_i$ for every $i \in I$. Prove that

$$\operatorname{Im}(Z^1(\mathfrak{U}, \mathbb{C}) \to Z^1(\mathfrak{V}, \mathbb{C}))$$

is a finite-dimensional vector space.

(c) Let X be a compact Riemann surface. Prove that $H^1(X, \mathbb{C})$ is a finite-dimensional vector space.

Hint. Use finite coverings $\mathfrak{U} = (U_i)$, $\mathfrak{V} = (V_i)$ of X with $V_i \Subset U_i$, such that all the U_i and V_i are isomorphic to disks.

Problem 35

Let X be a compact Riemann surface.

(a) Prove that the homomorphism

$$H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C}),$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{C}$, is injective.

(b) Show that $H^1(X, \mathbb{Z})$ is a finitely generated free $\mathbb{Z}$-module.

Hint. Show first, as in problem 34(c), that $H^1(X, \mathbb{Z})$ is finitely generated and then use (a) to prove that $H^1(X, \mathbb{Z})$ is free.

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p.t.o.

Problem 36*

Let $X := \mathbb{C}/\Lambda$ be a torus ($\Lambda \subset \mathbb{C}$ a lattice). Consider the following presheaf $\mathcal{F}$ on X (cf. Problem 21b): For $U \subset X$ open, let $\mathcal{F}(U)$ be the quotient group

$$\mathcal{F}(U) := \mathcal{O}^*(U)/\exp\mathcal{O}(U),$$

and for $V \subset U$ let $\varrho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ be the natural restriction homomorphism.

Prove that $\mathcal{F}$ does not satisfy sheaf axiom (Sh2) by explicitly constructing open subsets $U_1, U_2 \subset X$ and elements $f_\nu \in \mathcal{F}(U_\nu)$, $\nu = 1, 2$, which coincide on the intersection $V := U_1 \cap U_2$, i.e.

$$\varrho_V^{U_1}(f_1) = \varrho_V^{U_2}(f_2),$$

but there is **no** element $f \in \mathcal{F}(U)$, $U := U_1 \cup U_2$, with

$$\varrho_{U_1}^U(f) = f_1 \quad \text{and} \quad \varrho_{U_2}^U(f) = f_2.$$

Remark. This counter example shows that the assertion in Problem 23a) regarding sheaf axiom (Sh2) is not correct.

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*A correct, independently worked out solution of the starred problem is rewarded by a bonus of 0.3 in the final grading.

Please turn in your solution during the lecture or by email to

`forster@math.lmu.de`