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Riemann Surfaces

Problem sheet #8

Problem 29

Suppose $p_1, \ldots, p_n$ are pairwise distinct points of $\mathbb{C}$ and let

$$X := \mathbb{C} \smallsetminus \{p_1, \ldots, p_n\}.$$

Prove that $\quad H^1(X, \mathbb{Z}) \cong \mathbb{Z}^n$.

Hint. Construct an open covering $\mathfrak{U} = (U_1, U_2)$ of X such that the U_ν are connected and simply connected and $U_1 \cap U_2$ has $n+1$ connected components.

Problem 30

a) Show that $\mathfrak{U} = (\mathbb{P}^1 \smallsetminus \{\infty\}, \mathbb{P}^1 \smallsetminus \{0\})$ is a Leray covering of $\mathbb{P}^1$ for the sheaf Ω of holomorphic 1-forms on $\mathbb{P}^1$.

b) Prove that

$$H^1(\mathbb{P}^1, \Omega) \cong H^1(\mathfrak{U}, \Omega) \cong \mathbb{C}$$

and that the cohomology class of $\dfrac{dz}{z} \in \Omega(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \Omega)$ is a basis of $H^1(\mathbb{P}^1, \Omega)$.

Problem 31

Let X be the annulus $X := \{z \in \mathbb{C} : r < |z| < R\},\ 0 \leqslant r < R \leqslant \infty$.

a) Prove that for every $g \in \mathcal{E}(X)$ there exists an $f \in \mathcal{E}(X)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

b) Conclude that $H^1(X, \mathcal{O}) = 0$.

Problem 32

Let $\Lambda := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ be a lattice where $\tau \in \mathbb{C}$ with $T := \operatorname{Im}(\tau) > 0$. Let $X := \mathbb{C}/\Lambda$ be the associated torus and $p : \mathbb{C} \to X$ the canonical projection.

For real numbers $t_1 < t_2$ with $t_2 - t_1 \leqslant T$ define

$$Y(t_1, t_2) := \{z \in \mathbb{C} : t_1 < \operatorname{Im}(z) < t_2\}$$

and

$$U(t_1, t_2) := p(Y(t_1, t_2)) \subset X.$$

p.t.o.

a) Show that

$$\Phi : Y(t_1, t_2) \longrightarrow \mathbb{C}, \quad z \mapsto e^{2\pi i z},$$

maps $Y(t_1, t_2)$ onto the annulus

$$A(e^{-2\pi t_2}, e^{-2\pi t_1}) := \{z \in \mathbb{C} : e^{-2\pi t_2} < |z| < e^{-2\pi t_1}\}$$

and that there is a biholomorphic mapping $\varphi : U(t_1, t_2) \longrightarrow A(e^{-2\pi t_2}, e^{-2\pi t_1})$ which makes the following diagram commutative.

$$\begin{array}{ccc} Y(t_1, t_2) & \xrightarrow{\ p\ } & U(t_1, t_2) \\ & {\scriptstyle\Phi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & A(e^{-2\pi t_2}, e^{-2\pi t_1}) & \end{array}$$

b) Set $U_1 := U(0, T)$ and $U_2 := U(-T/2, T/2)$. Show that $\mathfrak{U} := (U_1, U_2)$ is a Leray covering of X for the sheaf $\mathcal{O}$ and that the intersection $U_1 \cap U_2$ consists of two connected components W_0 and W_1.

c) Prove that

$$H^1(X, \mathcal{O}) \cong \mathbb{C}$$

and that the function

$$f_0 \in \mathcal{O}(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \mathcal{O}) \quad \text{with} \quad f_0 \mid W_0 = 0 \text{ and } f_0 \mid W_1 = 1$$

represents a basis of $H^1(\mathfrak{U}, \mathcal{O}) \cong H^1(X, \mathcal{O})$.
