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Riemann Surfaces

Problem sheet #6

Problem 21

Let X be a Riemann surface.

a) For $U \subset X$ open, let $\mathcal{B}(U) \subset \mathcal{O}(U)$ be the vector space of all bounded holomorphic functions $f : U \to \mathbb{C}$.
Show that $\mathcal{B}$, together with the natural restriction maps, is a presheaf which satisfies sheaf axiom (Sh1), but not sheaf axiom (Sh2).

b) For $U \subset X$ open, define

$$\mathcal{F}(U) := \mathcal{O}^*(U)/\exp\mathcal{O}(U).$$

Show that $\mathcal{F}$ with the natural restriction maps is a presheaf (of multipicative abelian groups) which does not satisfy sheaf axiom (Sh1).

Problem 22

Let $(\mathcal{F}, \varrho)$ be a presheaf of abelian groups on a topological space X and $\mathcal{F}_x$, $x \in X$, its stalks. For an open subset $U \subset X$ define $\widetilde{\mathcal{F}}(U)$ to be the set of all families $(\varphi_x)_{x\in U}$ of elements $\varphi_x \in \mathcal{F}_x$ with the following property: For every $x \in U$ there exists an open neighborhood $V \subset U$ and an element $f \in \mathcal{F}(V)$ such that

$$\varphi_y = \varrho_y^V(f) \quad \text{for all } y \in V,$$

where $\varrho_y^V : \mathcal{F}(V) \to \mathcal{F}_y$ is the mapping that associates to $f \in \mathcal{F}(V)$ its germ in y.

a) Prove that, together with the natural restriction maps, $\widetilde{\mathcal{F}}$ is a sheaf on X.

b) For U open in X define $\alpha_U : \mathcal{F}(U) \to \widetilde{\mathcal{F}}(U)$ by $\alpha_U(f) := (\varrho_x^U(f))_{x\in U}$.
Show that the family (α_U) defines a presheaf homomorphism $\alpha : \mathcal{F} \to \widetilde{\mathcal{F}}$ which induces isomorphisms

$$\alpha_x : \mathcal{F}_x \to \widetilde{\mathcal{F}}_x \quad \text{for all } x \in X.$$

The sheaf $\widetilde{\mathcal{F}}$ is called the associated sheaf to $\mathcal{F}$.

Problem 23 (Continuation of problem 22)

a) Prove that the presheaf $\mathcal{F}$ satisfies sheaf axiom (Sh1) iff all $\alpha_U : \mathcal{F}(U) \to \widetilde{\mathcal{F}}(U)$ are injective, and sheaf axiom (Sh2) iff all α_U are surjective.

b) Let $\mathcal{S}$ be a sheaf on X and $\gamma : \mathcal{F} \to \mathcal{S}$ a presheaf homomorphism which induces isomorphisms $\gamma_x : \mathcal{F}_x \to \mathcal{S}_x$ for all $x \in X$. Prove that $\mathcal{S}$ is isomorphic to $\widetilde{\mathcal{F}}$.

c) Determine the associated sheaves of the presheaves of problem 21.

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p.t.o.

Problem 24

Let X, S be topological spaces, $p : X \to S$ a continuous map and $\mathcal{F}$ a sheaf of abelian groups on X. For $U \subset S$ open define

$$(p_*\mathcal{F})(U) := \mathcal{F}(p^{-1}(U)).$$

Show that $p_*\mathcal{F}$, together with the natural restriction maps, is a sheaf of abelian groups on S. It is called the *direct image sheaf* of $\mathcal{F}$ with respect to p.
