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May 11, 2016

# Riemann Surfaces

## Problem sheet #4

### Problem 13

Prove that every holomorphic map $f : \mathbb{P}_1 \to \mathbb{C}/\Lambda$ of the Riemann sphere to a torus is constant.

*Hint*. Use that $\mathbb{P}_1$ is simply connected.

### Problem 14

Let $X, Y, Z$ be locally compact Hausdorff spaces, $f : X \to Y$, $g : Y \to Z$ continuous maps and $h := g \circ f : X \to Z$ the composite map.

a) Which of the following implications are true, which are false?

i) $f$ and $g$ proper $\implies$ $h$ proper,

ii) $f$ and $h$ proper $\implies$ $g$ proper,

iii) $g$ and $h$ proper $\implies$ $f$ proper.

Give proofs or counter examples.

b) How does the situation change, if $f$ and $g$ are additionally supposed to be surjective?

### Problem 15

a) Show that every root $z \in \mathbb{C}$ of the polynomial

$$F(T) := T^n + a_1 T^{n-1} + \ldots + a_{n-1} T + a_n \in \mathbb{C}[T]$$

satisfies the estimate $|z| \leqslant 2 \max\{|a_k|^{1/k} : 1 \leqslant k \leqslant n\}$.

b) Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be the mapping defined by

$$\Phi(z_1, \ldots, z_n) := (s_k(z_1, \ldots, z_n))_{1 \leqslant k \leqslant n},$$

where $s_k$ are the elementary symmetric polynomials.

Prove that $\Phi$ is a proper, surjective map.

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**p.t.o.**

**Problem 16**

Let $p : Y \to X$ be an $n$-sheeted branched holomorphic covering map of compact Riemann surfaces $X$, $Y$. The trace map

$$\mathrm{Tr} = \mathrm{Tr}_{Y/X} : \mathcal{M}(Y) \to \mathcal{M}(X)$$

is defined as follows: For a meromorphic function $f \in \mathcal{M}(Y)$, let $\mathrm{Tr}(f)$ be the first elementary symmetric function of $f$ with respect to $p$, as defined in the course.

Show that all elementary symmetric functions of $f$ with respect to $p$ can be expressed polynomially in terms of $\mathrm{Tr}(f), \mathrm{Tr}(f^2), \ldots, \mathrm{Tr}(f^n)$.

Give explicit formulas in the cases $n = 2$ and $n = 3$.

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