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April 27, 2016

# Riemann Surfaces

## Problem sheet #2

### Problem 5

For an open subset $U \subset X$ of a Riemann surface $X$ we denote by $\mathcal{O}_X(U)$ or briefly $\mathcal{O}(U)$ the ring of all holomorphic functions $f : U \to \mathbb{C}$.

Let $X$ and $Y$ be two Riemann surfaces and $\Phi : X \to Y$ a continuous mapping. Prove that $\Phi$ is holomorphic if and only if for every open subset $V \subset Y$ and every $f \in \mathcal{O}_Y(V)$ the function

$$f \circ \Phi : \Phi^{-1}(V) \longrightarrow \mathbb{C}$$

belongs to $\mathcal{O}_X(\Phi^{-1}(V))$. In this case the mapping

$$\Phi^* : \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(\Phi^{-1}(V)), \qquad f \mapsto f \circ \Phi,$$

is a ring homomorphism.

### Problem 6

a) Let $U \subset \mathbb{R}^2 \mathrel{\widehat{=}} \mathbb{C}$ be an open subset and $f : U \to \mathbb{R}$ be a harmonic function, i.e. 2-times continuously differentiable and satisfying the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Let $V \subset \mathbb{C}$ be another open subset and $\varphi : V \to U$ be a biholomorphic mapping. Prove that the composite function $f \circ \varphi : V \to \mathbb{R}$ is also harmonic.

*Remark.* This implies that one can define the notion of harmonic function on a Riemann surface.

b) Let $X$ be a Riemann surface and $f : X \to \mathbb{R}$ a non-constant harmonic function. Show that $f$ does not attain its maximium. In particular, every harmonic function $f : X \to \mathbb{R}$ on a compact Riemann surface is constant.

### Problem 7

Let $p_1, \ldots, p_n \in X$ be points on a compact Riemann surface $X$ and let

$$X' := X \setminus \{p_1, \ldots, p_n\}.$$

(For example $X = \mathbb{P}_1$ and $X' = \mathbb{C}$.)

Suppose that $f : X' \to \mathbb{C}$ is a holomorphic function and $W \subset \mathbb{C}$ a non-empty open subset with $f(X') \subset \mathbb{C} \setminus W$. Prove that $f$ is constant.

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**p.t.o.**

**Problem 8**

Let $p_1, \dots, p_n \in X$ be points on a compact Riemann surface $X$ and let

$$X' := X \setminus \{p_1, \dots, p_n\}.$$

a) Prove that every automorphism of $X'$ (i.e. biholomorphic map onto itself) extends to an automorphism of $X$.

b) Using a), determine all automorphisms of $\mathbb{C}^*$ and of $X := \mathbb{C} \setminus \{0, 1\}$.

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