Mathematisches Institut
der Universität München
Prof. Otto Forster

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Riemann Surfaces

Problem sheet #1

Problem 1

Let X be a Riemann surface whose complex structure is defined by an atlas

$$\mathfrak{A} := \{\varphi_j : U_j \to V_j \,|\, j \in J\}.$$

Denote by $\sigma : \mathbb{C} \to \mathbb{C}$ the complex conjugation. Define $\mathfrak{A}^\sigma$ as the set of all complex charts

$$\sigma \circ \varphi_j : U_j \to \sigma(V_j) \subset \mathbb{C}, \qquad j \in J.$$

a) Prove that $\mathfrak{A}^\sigma$ is again a complex atlas on the topological space underlying X, and thus defines a Riemann surface which will be denoted by X^σ.

b) Show that the atlas $\mathfrak{A}^\sigma$ is not holomorphically equivalent with $\mathfrak{A}$, but there exist Riemann surfaces X which are isomorphic to X^σ (i.e. there exists a biholomorphic map $\varphi : X \to X^\sigma$).

Problem 2

Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$,

$$\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and let $N := (0, 0, 1)$ be the north pole of $\mathbb{S}^2$. We identify the plane $\{x_3 = 0\} \subset \mathbb{R}^3$ with the complex number plane $\mathbb{C}$ by the correspondence $(x_1, x_2, 0) \mapsto x_1 + ix_2$.
The *stereographic projection*

$$\mathrm{st} : \mathbb{S}^2 \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{P}^1$$

is defined as follows: For $x \in \mathbb{S}^2 \setminus \{N\}$ let $\mathrm{st}(x)$ be the intersection of the plane $\{x_3 = 0\}$ with the line through N and x. For the north pole one defines $\mathrm{st}(N) := \infty$.

a) Show that the stereographic projection st is given by the formula

$$\mathrm{st}(x) = \frac{1}{1 - x_3}(x_1 + ix_2) \qquad \text{for all } x \in \mathbb{S}^2 \setminus \{N\}.$$

b) An element A of the special orthogonal group

$$SO(3) = \{A \in GL(3, \mathbb{R}) : A^T A = E,\ \det A = 1\}$$

definies a bijective map of the sphere $\mathbb{S}^2$ onto itself.

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p.t.o.

Prove that the map

$$f := \mathrm{st} \circ A \circ \mathrm{st}^{-1} : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

is biholomorphic.

Hint. Use the fact that the group $SO(3)$ is generated by the subgroup of rotations with axis $\mathbb{R}(0,0,1)$ and the special transformation $(x_1, x_2, x_3) \mapsto (x_1, x_3, -x_2)$.

c) Do all biholomorphic maps $\mathbb{P}^1 \to \mathbb{P}^1$ arise in this way?

Problem 3

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ and $\Lambda' = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$ be two lattices in $\mathbb{C}$. Show that $\Lambda = \Lambda'$ if and only if there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) = \{A \in M(2 \times 2, \mathbb{Z}) : \det A = \pm 1\}$$

such that

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Problem 4

a) Let $\Lambda, \Lambda' \subset \mathbb{C}$ be two lattices. Let $\alpha \in \mathbb{C}^*$ be a complex number such that $\alpha\Lambda \subset \Lambda'$. Show that the map $\mathbb{C} \to \mathbb{C}$, $z \mapsto \alpha z$, induces a holomorphic map

$$\phi_\alpha : \mathbb{C}/\Lambda \longrightarrow \mathbb{C}/\Lambda',$$

which is biholomorphic if and only if $\alpha\Lambda = \Lambda'$.

b) Show that every torus $\mathbb{C}/\Lambda$ is isomorphic to a torus of the form

$$X(\tau) := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

with $\tau \in \mathbb{H}$, where $\mathbb{H}$ denotes the upper halfplane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$.

c) Suppose $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$. Let

$$\tau' := \frac{a\tau + b}{c\tau + d}.$$

Prove that $\mathrm{Im}(\tau') > 0$ and the tori $X(\tau)$ and $X(\tau')$ are isomorphic.
