Mathematisches Institut
der Universität München
Prof. Otto Forster

WS 2012/13
January 10, 2013

Riemann Surfaces

Problem sheet #10

Problem 37

On a compact Riemann surface X let $\mathcal{Q} \subset \mathcal{M}^{(1)}$ be the sheaf of meromorphic 1-forms which have residue 0 at every pole.

a) Show that $\quad 0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{M} \xrightarrow{d} \mathcal{Q} \longrightarrow 0 \quad$ is a short exact sequence of sheaves.

b) Prove that there is an isomomorphism

$$H^1(X, \mathbb{C}) \cong \mathcal{Q}(X)/d\mathcal{M}(X).$$

Problem 38

Let $\Lambda \subset \mathbb{C}$ be a lattice. Prove that on the torus $X := \mathbb{C}/\Lambda$ a basis of the vector space $\mathcal{Q}(X)/d\mathcal{M}(X)$ is given by the classes of the differential forms

$$\sigma_1 := dz \quad \text{and} \quad \sigma_2 := \wp_\Lambda dz$$

modulo $d\mathcal{M}(X)$.

Problem 39

Let X and Y be compact Riemann surfaces. Suppose X has genus 1 and

$$F : X \to Y$$

is a non-constant holomorphic map.

Prove that *either* $Y \cong \mathbb{P}^1$, *or* Y has genus 1 and F is an unbranched covering map.

Problem 40

Let X be a compact Riemann surface of genus $g > 0$ and let $p \in X$ be a point. Prove:

a) For every $n \geqslant 2g$ there exists a meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order n at p and is holomorphic in $X \smallsetminus \{p\}$.

b) There exist precisely g "*lacunary exponents*"

$$1 = n_1 < n_2 < \cdots < n_g < 2g$$

such that there is *no* meromorphic function $f \in \mathcal{M}(X)$ which has a pole of order n_i at p and is holomorphic in $X \smallsetminus \{p\}$.

Due: Wednesday, January 23, 2013, 15 h