Mathematisches Institut
der Universität München
Prof. Otto Forster

WS 2012/13
December 19, 2012

Riemann Surfaces

Problem sheet #9

Problem 33

Let D be a divisor on $\mathbb{P}_1$. Prove that

$$\dim H^1(\mathbb{P}_1, \mathcal{O}_D) = \max(0, -1 - \deg D).$$

Problem 34

a) Show that $\mathfrak{U} = (\mathbb{P}_1 \setminus \{\infty\}, \mathbb{P}_1 \setminus \{0\})$ is a Leray covering of $\mathbb{P}_1$ for the sheaf Ω of holomorphic 1-forms on $\mathbb{P}_1$.

b) Prove that

$$H^1(\mathbb{P}_1, \Omega) \cong H^1(\mathfrak{U}, \Omega) \cong \mathbb{C}$$

and that the cohomology class of $\dfrac{dz}{z} \in \Omega(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \Omega)$ is a basis of $H^1(\mathbb{P}_1, \Omega)$.

Problem 35 Let X be the annulus $X := \{z \in \mathbb{C} : r < |z| < R\}$, $0 \leqslant r < R \leqslant \infty$.

a) Prove that for every $g \in \mathcal{E}(X)$ there exists an $f \in \mathcal{E}(X)$ such that

$$\frac{\partial f}{\partial \bar{z}} = g.$$

b) Conclude that $H^1(X, \mathcal{O}) = 0$.

Problem 36

Let $q \in \mathbb{C}$ with $r := |q| > 1$ and let G be the multiplicative group $G := \{q^n : n \in \mathbb{Z}\} \subset \mathbb{C}^*$. Consider the Riemann surface $X := \mathbb{C}^*/G$, cf. Problem 8. Let $\pi : \mathbb{C}^* \to X$ be the canonical projection.

a) Let $Y_1 := \{z \in \mathbb{C} : 1 < |z| < r\}$ and $Y_2 := \{z \in \mathbb{C} : \varrho < |z| < \varrho r\}$, where $1 < \varrho < r$, and $U_\nu := \pi(Y_\nu)$. Show that $\pi \mid Y_\nu \to U_\nu$ is biholomorphic and that $\mathfrak{U} := (U_1, U_2)$ is a Leray covering of X for the sheaf $\mathcal{O}$. The intersection $U_1 \cap U_2$ consists of two connected components V_1 and V_2.

b)* Let $f_0 \in \mathcal{O}(U_1 \cap U_2)$ be the function with $f_0 \mid V_1 = 0$ and $f_0 \mid V_2 = 1$. Prove that

$$H^1(X, \mathcal{O}) \cong H^1(\mathfrak{U}, \mathcal{O}) \cong \mathbb{C}$$

and that the cohomology class of $f_0 \in \mathcal{O}(U_1 \cap U_2) \cong Z^1(\mathfrak{U}, \mathcal{O})$ is a basis of $H^1(X, \mathcal{O})$.

Due: Wednesday, January 16, 2013, 15 h