Mathematisches Institut
der Universität München
Prof. Otto Forster

WS 2012/13
November 30, 2012

Riemann Surfaces

Problem sheet #7

Problem 25

(Cf. problem 19) Let $p : X_3 \to \mathbb{P}_1$ be the Riemann surface of $\sqrt[3]{1-z^3}$, i.e. of the algebraic function defined by the polynomial

$$w^3 + z^3 - 1 \in \mathcal{M}(\mathbb{P}_1)[w], \qquad \mathcal{M}(\mathbb{P}_1) \cong \mathbb{C}(z).$$

a) Determine all zeros and poles of the differential form dz on X_3.

b) Prove that the differential form $\omega := dz/w^2$ is holomorphic on X_3 and has no zeros.

Problem 26

Let X be a Riemann surface. For $Y \subset X$ open, the conjugation $\operatorname{conj} : \mathcal{E}^{(1)}(Y) \to \mathcal{E}^{(1)}(Y)$ is defined as follows: With respect to a local coordinate neighborhood (U, z) a differential form ω can be written as $\omega = f dz + g d\bar{z}$. Then $\operatorname{conj}(\omega) := \bar{f} d\bar{z} + \bar{g} dz$.

a) Show that this definition is independent of the local coordinate and thus conj is well-defined. One writes briefly $\bar{\omega}$ for $\operatorname{conj}(\omega)$.

b) Prove the following formulas for $g \in \mathcal{E}(Y)$, $\omega \in \mathcal{E}^{(1)}(Y)$:

$$\overline{dg} = d\bar{g}, \quad \overline{d'g} = d''\bar{g}, \quad \overline{d''g} = d'\bar{g}, \quad \overline{g\omega} = \bar{g}\bar{\omega}$$

c) If $c : [0,1] \to Y$ is a piecewise differentiable curve and $\omega \in \mathcal{E}^{(1)}(Y)$, then $\overline{\int_c \omega} = \int_c \bar{\omega}$.

Problem 27

a) Let X be a Riemann surface and $U \subset X$ open. Prove that a function $h \in \mathcal{E}(U)$ is harmonic if and only if the differential form $d'h$ is holomorphic.

b) Let $h : X \to \mathbb{R}$ be a real harmonic function. Prove that all periods of the differential form $\omega := d'h \in \Omega(X)$ are purely imaginary.

c) For h and ω as in b), prove that h is the real part of a holomorphic function $f : X \to \mathbb{C}$ if and only if all periods of ω vanish.

Problem 28

Let $X := \{z \in \mathbb{C} : r < |z| < R\}$, $0 \leqslant r < R \leqslant \infty$, and let $u : X \to \mathbb{R}$ be a harmonic function. Using 27c), prove that there is a constant $c \in \mathbb{R}$ and a holomorphic function $f : X \to \mathbb{C}$ such that

$$u(z) = c \log |z| + \operatorname{Re}(f(z)) \qquad \text{for all } z \in X.$$

Due: Wednesday, December 12, 2012, 15 h