Mathematisches Institut
der Universität München
Prof. Otto Forster

WS 2012/13
November 16, 2012

Riemann Surfaces

Problem sheet #5

Problem 17

a) Show that every root $z \in \mathbb{C}$ of the polynomial

$$F(T) := T^n + a_1 T^{n-1} + \ldots + a_{n-1}T + a_n \in \mathbb{C}[T]$$

satisfies the estimate $|z| \leqslant 2\max\{|a_k|^{1/k} : 1 \leqslant k \leqslant n\}$.

b) Let $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ be the mapping defined by

$$\Phi(z_1, \ldots, z_n) := (\sigma_k(z_1, \ldots, z_n))_{1 \leqslant k \leqslant n},$$

where σ_k are the elementary symmetric polynomials.
Prove that Φ is a proper, surjective map.

Problem 18

Let $p : X \to S$ be an n-sheeted branched holomorphic covering map of compact Riemann surfaces X, S. The trace map

$$\mathrm{Tr} = \mathrm{Tr}_{X/S} : \mathcal{M}(X) \to \mathcal{M}(S)$$

is defined as follows: For a function $f \in \mathcal{M}(X)$, let $\mathrm{Tr}(f)$ be the first elementary symmetric function of f with respect to p, as defined in the course.
Show that all elementary symmetric functions of f with respect to p can be expressed polynomially in terms of $\mathrm{Tr}(f), \mathrm{Tr}(f^2), \ldots, \mathrm{Tr}(f^n)$. Give explicit formulas in the cases $n = 2$ and $n = 3$.

Problem 19

Let $p : X_3 \to \mathbb{P}_1$ be the Riemann surface of $\sqrt[3]{1 - z^3}$, i.e. of the algebraic function defined by the polynomial

$$w^3 + z^3 - 1 \in \mathcal{M}(\mathbb{P}_1)[w], \qquad \mathcal{M}(\mathbb{P}_1) \cong \mathbb{C}(z).$$

Determine all branch points and critical values of p and show that there are no branch points above $\infty \in \mathbb{P}_1$.

Problem 20

On the compact Riemann surface X_3 defined in Problem 19 consider the function $f := w + z$.

a) Determine the poles and zeros of f.

b) Calculate the elementary symmetric functions of f with respect to $p : X_3 \to \mathbb{P}_1$.

Due: Wednesday, November 28, 2012, 15 h