Mathematisches Institut
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WS 2012/13
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Riemann Surfaces

Problem sheet #3

Problem 9

Let $f\colon \mathbb{P}_1 \to \mathbb{P}_1$ be the holomorphic map defined by the rational function $f(z) := z + \dfrac{1}{z}$.
Determine the branch points of f and show that there are biholomorphic maps $\varphi : \mathbb{P}_1 \to \mathbb{P}_1$ and $\psi : \mathbb{P}_1 \to \mathbb{P}_1$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{P}_1 & \xrightarrow{\varphi} & \mathbb{P}_1 \\ f\downarrow & & \downarrow p_2 \\ \mathbb{P}_1 & \xrightarrow{\psi} & \mathbb{P}_1 \end{array}$$

Here $p_2 : \mathbb{P}_1 \to \mathbb{P}_1$ is the map $z \mapsto p_2(z) := z^2$.

Problem 10

a) Show that the tangent function defines a locally biholomorphic map $\tan : \mathbb{C} \to \mathbb{P}_1$.

b) Prove that $\tan(\mathbb{C}) = \mathbb{P}_1 \smallsetminus \{\pm i\}$ and that $\tan : \mathbb{C} \to \mathbb{P}_1 \smallsetminus \{\pm i\}$ is a covering map.

c) Show that there exist biholomorphic maps

$$\varphi : \mathbb{C} \to \mathbb{C} \qquad \text{and} \qquad \psi : \mathbb{C}^* \to \mathbb{P}_1 \smallsetminus \{\pm i\}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\varphi} & \mathbb{C} \\ \exp\downarrow & & \downarrow \tan \\ \mathbb{C}^* & \xrightarrow{\psi} & \mathbb{P}_1 \smallsetminus \{\pm i\} \end{array}$$

Problem 11

Let X, Y, Z be locally compact Hausdorff spaces, $f : X \to Y$, $g : Y \to Z$ continuous maps and $h := g \circ f : X \to Z$ the composite map.

Which of the following implications are true, which are false?

i) f and g proper $\implies$ h proper,
ii) f and h proper $\implies$ g proper,
iii) g and h proper $\implies$ f proper.

Give proofs or counter examples.

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p.t.o.

Problem 12

Let $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$, ($\omega_1, \omega_2 \in \mathbb{C}$ linearly independent over $\mathbb{R}$), be a lattice. The Weierstrass $\wp$-function with respect to Λ is defined by

$$\wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\omega \in \Lambda \smallsetminus 0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big).$$

a) Prove that for every compact disc $K_r := \{z \in \mathbb{C} : |z| \leqslant r\}$ there exists a finite subset $\Lambda_0 \subset \Lambda$ such that $\omega \notin K_r$ for all $\omega \in \Lambda \smallsetminus \Lambda_0$ and the series

$$\sum_{\omega \in \Lambda \smallsetminus \Lambda_0} \Big(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \Big).$$

converges uniformly on K_r. This implies that $\wp_\Lambda$ is a meromorphic function on $\mathbb{C}$ with poles of order two exactly at the lattice points $\omega \in \Lambda$.

b) Show that $\wp_\Lambda$ a doubly periodic meromorphic function on $\mathbb{C}$ with respect to Λ, i.e. $\wp_\Lambda(z) = \wp_\Lambda(z+\omega)$ for all $\omega \in \Lambda$ and all $z \in \mathbb{C}$.
Hint. Prove first that the derivative $\wp'_\Lambda(z) = -2 \sum_{\omega \in \Lambda} \frac{1}{(z-\omega)^3}$ is doubly periodic.

c) Since $\wp_\Lambda$ is periodic with respect to Λ, it defines a holomorphic map $\mathbb{C}/\Lambda \to \mathbb{P}_1$. Prove that this map is a two-sheeted branched covering map with exactly 4 branch points at

$$[0], \Big[\frac{\omega_1}{2}\Big], \Big[\frac{\omega_2}{2}\Big], \Big[\frac{\omega_1 + \omega_2}{2}\Big] \in \mathbb{C}/\Lambda.$$

Hint. To determine the zeros of $\wp_\Lambda$, use that $\wp'_\Lambda$ is an odd function of z, i.e. $\wp'_\Lambda(-z) = -\wp'_\Lambda(z)$ for all $z \in \mathbb{C}$.

Due: Wednesday, November 14, 2012, 15 h