Mathematisches Institut
der Universität München
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WS 2012/13
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Riemann Surfaces

Problem sheet #1

Problem 1 Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$,

$$\mathbb{S}^2 := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}$$

and let $N := (0, 0, 1)$ be the north pole of $\mathbb{S}^2$. We identify the plane $\{x_3 = 0\} \subset \mathbb{R}^3$ with the complex number plane $\mathbb{C}$ by the correspondence $(x_1, x_2, 0) \mapsto x_1 + ix_2$.
We define a map $\mathrm{st} : \mathbb{S}^2 \to \mathbb{C} \cup \{\infty\} = \mathbb{P}_1$ (*stereographic projection*) in the following way: For $x \in \mathbb{S}^2 \smallsetminus \{N\}$ we let $\mathrm{st}(x)$ be the intersection of the plane $\{x_3 = 0\}$ with the line through N and x. For the north pole we define $\mathrm{st}(N) := \infty$.
Show that $\mathrm{st} : \mathbb{S}^2 \to \mathbb{P}_1(\mathbb{C})$ is a homeomorphism and one has

$$\mathrm{st}(x) = \frac{1}{1 - x_3}(x_1 + ix_2) \qquad \text{for all } x \in \mathbb{S}^2 \smallsetminus \{N\}.$$

Problem 2 Let $\mathrm{st} : \mathbb{S}^2 \to \mathbb{P}_1(\mathbb{C})$ be as in Problem 1. An element A of the special orthogonal group

$$SO(3) = \{A \in GL(3, \mathbb{R}) : A^T A = E,\ \det A = 1\}$$

definies a bijective map of the sphere $\mathbb{S}^2$ onto itself. Prove: The map

$$f := \mathrm{st} \circ A \circ \mathrm{st}^{-1} : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$$

is biholomorphic.
Do all biholomorphic maps $\mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$ arise in this way?
Hint. Use the fact that the group $SO(3)$ is generated by the subgroup of rotations with axis $\mathbb{R}(0, 0, 1)$ and the special transformation $(x_1, x_2, x_3) \mapsto (x_3, x_2, -x_1)$.

Problem 3 Let X be a Riemann surface, whose complex structure is defined by an atlas

$$\mathfrak{A} := \{\varphi_j : U_j \to V_j \,|\, j \in J\}.$$

Denote by $\sigma : \mathbb{C} \to \mathbb{C}$ the complex conjugation. Define $\mathfrak{A}^\sigma$ as the set of all complex charts

$$\sigma \circ \varphi_j : U_j \to \sigma(V_j) \subset \mathbb{C}, \qquad j \in J.$$

a) Show that $\mathfrak{A}^\sigma$ is again a complex atlas on the topological space underlying X, and thus defines a Riemann surface which will be denoted by X^σ.
b) Show that the Riemann surface $\mathbb{P}_1(\mathbb{C})^\sigma$ is isomorphic to $\mathbb{P}_1(\mathbb{C})$.

Problem 4 For $\tau \in \mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ let E_τ be the torus $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$.
Prove: The Riemann surface $(E_\tau)^\sigma$ is isomorphic to a torus $E_{\tau'}$ with $\tau' \in \mathbb{H}$. Calculate τ' as a function of τ.

Due: Wednesday, October 31, 2012, 15 h