MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Jan. 24, 2005

Algebraic Number Theory

Problem Sheet #12

Problem 45

a) Decompose the polynomial $\overline{\Phi}_7(X) = \sum_{k=0}^{6} X^k \in \mathbb{F}_{29}[X]$ into a product of linear factors.

b) Write 29 as a product of six prime elements of the ring $\mathbb{Z}[e^{2\pi i/7}]$.

Problem 46

Let K be the cyclotomic field $K = \mathbb{Q}(\zeta)$, $\zeta := e^{2\pi i/p}$, where p is an odd prime.

Prove that $\mathbb{Q}(\sqrt{p}) \subset K$ for $p \equiv 1 \bmod 4$, and $\mathbb{Q}(\sqrt{-p}) \subset K$ for $p \equiv 3 \bmod 4$.

Hint. Show that the element

$$S := \sum_{k=1}^{p-1} \left(\frac{k}{p}\right) \zeta^k \in \mathbb{Q}(\zeta)$$

satisfies $S^2 = (-1)^{(p-1)/2} p$.

Problem 47

Let $K := \mathbb{Q}(\sqrt{-7}) \subset L := \mathbb{Q}(e^{2\pi i/7})$. Show that $\mathfrak{o}_L$ is a free $\mathfrak{o}_K$-module of rank 3, i.e. there exist elements $\omega_1, \omega_2, \omega_3 \in \mathfrak{o}_L$, linearly independent over K, such that

$$\mathfrak{o}_L := \sum_{j=1}^{3} \mathfrak{o}_K \cdot \omega_j.$$

Determine explicitly such elements $\omega_1, \omega_2, \omega_3$.

Problem 48

Consider the four rings

$$\mathbb{Z}[\sqrt{-7}] \subset \mathbb{Z}[\tfrac{1+\sqrt{-7}}{2}] \subset \mathbb{Z}[\tfrac{1}{2}, \sqrt{-7}] \subset \mathbb{Q}[\sqrt{-7}].$$

Which of them are Dedekind rings?

Due: Tuesday, February 1, 2005, 14:10 h

The Final Written Exam (Klausur) will be held on
Thursday, February 3, 2005, 11 o'clock, in room E05