MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2004/2005
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Algebraic Number Theory

Problem Sheet #11

Problem 41

Let K be an algebraic number field of degree n and $\Lambda \subset K$ a lattice, i.e.

$$\Lambda = \mathbb{Z}a_1 + \mathbb{Z}a_2 + \ldots + \mathbb{Z}a_n$$

with $\mathbb{Q}$-linearly independent elements $a_1, \ldots, a_n \in K$. The discriminant of Λ is defined as the discriminant of $(a_1, \ldots, a_n)$ (which is independent of the choice of the $\mathbb{Z}$-basis $a_1, \ldots, a_n$ of Λ).

Prove that for every $x \in K^*$ the following formula holds:

$$\operatorname{discr}(x\Lambda) = \mathrm{N}(x)^2 \operatorname{discr}(\Lambda).$$

Problem 42

Let p be a prime number, $n \geq 1$ an integer and

$$\Phi_{p^n}(X) := \sum_{k=0}^{p-1} X^{p^{n-1}k} \in \mathbb{Q}[X].$$

a) Show that $\Phi_{p^n}(X)$ is irreducible over $\mathbb{Q}$ and is the minimal polynomial of the p^n-th root of unity $\zeta := e^{2\pi i/p^n}$.

b) For the field extension $\mathbb{Q} \subset \mathbb{Q}(\zeta)$, calculate the norms and traces of the elements

$$\zeta^k,\ k \geq 0, \quad \text{and} \quad 1 - \zeta.$$

Problem 43

With $\zeta = e^{2\pi i/p^n}$ as in problem 42, show that $\mathbb{Z}[\zeta]$ is the ring of integers in the algebraic number field $\mathbb{Q}(\zeta)$.

Problem 44

Let $K := \mathbb{Q}(\sqrt{5})$ and $L := \mathbb{Q}(\zeta_5)$, where $\zeta_5 := e^{2\pi i/5}$.

a) Show that $K \subset L$ and $\mathfrak{o}_K \subset \mathfrak{o}_L$.

b) For $p = 2, 3, 5, 11, 19$, decompose p as

(i) a product of prime elements of $\mathfrak{o}_K$,

(ii) a product of prime elements of $\mathfrak{o}_L$

and compare the decompositions.

Due: Tuesday, January 25, 2005, 14:10 h