MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
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WS 2004/2005
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Algebraic Number Theory

Problem Sheet #10

Problem 37

Let $f(X)$ be the quadratic polynomial

$$f(X) = X^2 + X + q \in \mathbb{Z}[X],$$

where q is an integer ≥ 3.

a) Suppose that $f(n)$ is prime for all integers n with $0 \leq n < \sqrt{q/3}$. Show that the field $\mathbb{Q}(\sqrt{1-4q})$ has class number 1.

b) Deduce from a) and problem 36:

If $f(n)$ is prime for all integers n with $0 \leq n < \sqrt{q/3}$, then $f(n)$ is prime for all integers n with $0 \leq n \leq q-2$.

Problem 38

Let $\mathbb{Q}(\theta)$ be a cubic number field, where θ is a root of the irreducible polynomial

$$f(X) = X^3 + aX + b \in \mathbb{Q}[X].$$

Calculate the discriminant $\operatorname{discr}(1, \theta, \theta^2)$.

Problem 39

a) Determine the ring of integers in the algebraic number field $\mathbb{Q}(\cos \frac{2\pi}{7})$.
(Cf. problem 3).

b) Prove that $\mathbb{Q}(\cos \frac{2\pi}{7})$ is Galois over $\mathbb{Q}$.

Problem 40

In $\mathbb{Z}^3$ consider the sublattice

$$\Lambda := \{(x, y, z) \in \mathbb{Z}^3 : x + y \equiv 0 \bmod 2,\ x + 2y + 3z \equiv 0 \bmod 5\}.$$

Determine a $\mathbb{Z}$-basis of Λ, i.e. linearly independent elements $v_1, v_2, v_3 \in \Lambda$ with

$$\Lambda = \mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3.$$

Due: Tuesday, January 18, 2005, 14:10 h