MATHEMATISCHES INSTITUT
DER UNIVERSITÄT MÜNCHEN
Prof. Otto Forster

WS 2004/2005
Dec. 20, 2004

Algebraic Number Theory
Problem Sheet #9

Problem 33

For the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$, calculate the discriminant

$$\operatorname{discr}(1, \sqrt[3]{2}, \sqrt[3]{4}).$$

Problem 34

For the field extension $\mathbb{Q} \subset \mathbb{Q}(\sqrt[3]{2})$, calculate the norm

$$N(x + y\sqrt[3]{2} + z\sqrt[3]{4}), \qquad x, y, z \in \mathbb{Q}.$$

Problem 35

Let $p \geq 7$ be a squarefree integer with $p \equiv 3 \bmod 4$. Define the integer q by $p = 4q - 1$.

a) Show that the norm of an element $\xi = x + \frac{1+\sqrt{-p}}{2} \in \mathbb{Q}(\sqrt{-p})$ is given by

$$N(\xi) = x^2 + x + q.$$

In the following, suppose that $K = \mathbb{Q}(\sqrt{-p})$ has class number 1.

b) Prove that q is prime.

Hint. If $\ell < q$ is a prime with $\ell \mid q$, then $\left(\ell, \dfrac{1+\sqrt{-p}}{2}\right)_{\mathbb{Z}}$ is a non-principal ideal of $\mathfrak{o}_K$.

c) Prove that p is prime.

Hint. If p is not prime, construct a reduced ideal $\left(a, \dfrac{b+\sqrt{-p}}{2}\right)_{\mathbb{Z}} \subset \mathfrak{o}_K$ which is not the unit ideal.

Problem 36*

Let p, q be as in problem 35 and suppose that $K = \mathbb{Q}(\sqrt{-p})$ has class number 1.

a) Prove that every odd prime $\ell < q$ is inert in $\mathfrak{o}_K$.

b) Prove that the polynomial

$$f(x) := x^2 + x + q$$

takes prime values for $x = 0, 1, 2, \dots, q - 2$.

Due: Tuesday, January 11, 2005, 14:10 h

Merry Christmas and a Happy New Year!